LIMITS

Dr. Jennifer Bergner, Dr. Don Spickler
Salisbury University

Introduction and Goals

You have been exploring limits in class and homework by a variety of approaches. In this lab we will look at exploring limits using tables, graphs, and a new Maple command.

Before you start

You have been examining limits of functions in class and homework. Make sure to take a look over sections 2.2 - 2.3 before you start this lab. Remember, Maple is a tool and provides results that you will have to interpret. At times, some of the maple output may seem to contradict itself or what you know. If this happens, explore!

Textbook Correspondence

Stewart 5th edition: 2.2-2.3

Maple Commands and Packages Used

Packages: none

Commands: limit, map, Limit, evalf, seq

Skills: defining a list and using the sequence command

Maple Commands

Creating a list

One thing your graphing calculator can do is generate a list or table of values. This list could contain the input values that we wish to plug into a given function. We can also create lists in Maple. A list in Maple is an ordered sequence of expressions enclosed in square brackets. It is important you use square brackets, this is your way of telling Maple that you care about the order you put the list’s elements in. It is also good to always name things in Maple. So choose a name for your list.

Example

Say I wanted to create the list that contained (in order) the values 0.01, 0.001, 0.0001 and I wanted to name it “future”. I would type:

```maple
> future:=[0.01, 0.001, 0.0001];
future := [0.01, 0.001, 0.0001]
```
The map command

Maple has a nice command named *map*, that will take values from a list and evaluate a function at each of those values, returning the output in an ordered list. This is what you do when you are computing values of a function to put in a table of values:

<table>
<thead>
<tr>
<th>x</th>
<th>-3.1</th>
<th>-1</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>4</th>
<th>10</th>
<th>11.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-1.1</td>
<td>1</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>13.5</td>
</tr>
</tbody>
</table>

After you have defined the function $f(x)$ in Maple and put your input values into a list, the general syntax (the way you communicate to Maple) is:

```maple
map(f,list);
```

This is how it works:

First define the function.

```maple
> f:=x->x+2;
f := x → x + 2
```

Second, create a list with the input we wish to plug in. I will call the list “sample”.

```maple
> sample:=[-1.1,-1,0,0.5,1,4,10,11.5];
sample := [-1.1, -1, 0, 0.5, 1, 4, 10, 11.5]
```

Finally, issue the map command:

```maple
> map(f,sample);
[0.9, 1, 2, 2.5, 3, 6, 12, 13.5]
```

Here you see we get the output values that belong to the input values that were in the list. This is the second row of your table of values.

The sequence command

Just as there is a quick way to find the output values using Maple’s map command, there is a quick way to generate a list of input values.

To generate a list of input values that are closer and closer to 0 from the right, we type

```maple
> [seq(1/(10^i),i=1..10)];
```
This is your way of telling Maple to generate a list (or sequence) of 10 values that are generated by the formula \( \frac{1}{10^i} \) by letting \( i = 1 \) then 2 then 3, etc… until \( i = 10 \). This will give you the ten values \( \frac{1}{10}, \frac{1}{100} \) and so on with the last being \( \frac{1}{10000000000} \).

To name this list “rlist”, we type:

```maple
> rlist:=[seq(1/(10^i),i=1..10)];
```

If you wanted to get values closer and closer to 0 from the left, the same command works with one additional change:

```maple
> llist:=[seq(-1/(10^i),i=1..10)];
```

**Finding limits with Maple as a tool**

As you have learned from class and homework exercises, there are primarily three ways to find a limit: numerically, graphically, and algebraically. Maple also gives a fourth way that can be called the "symbolic" method.

**Method 1.** To compute \( \lim_{x \to a} f(x) \) with the numerical or tabular method, one inputs values of \( x \) that are successively closer to the value of \( a \) and examine the corresponding outputs to see if they are approaching a common value. This may be tedious by hand, but a computer algebra system will aid tremendously.

The map command can help us compute limits using method 1. Suppose we wish to compute \( \lim_{x \to 1} f(x) \) where \( f(x) = 3x^2 - 5 \). We need to examine what functional values belong to input that is getting closer and closer to \( x = 1 \). One way to get close to 1 is to come in from the right: 1.2, 1.1, 1.01, 1.001, 1.0001, 1.00001. Another way to get close to 1 is to come in from the left: 0.9, 0.99, 0.999, 0.999999, 0.999999999999. If we could plug each of these values into \( f(x) \) and check out where (if anywhere) the output is going we could evaluate the limit.

First, define our function

```maple
> f:=x->3*x^2-5;
```

Second, define our list of values from the right and left. I will call them rightlist and leftlist.

```maple
rightlist:=[1.2,1.1,1.01,1.001,1.0001,1.00001];
leftlist:=[0.9,0.99,0.999,0.99999,0.9999999999];
```

Notice how the list name is set equal to (:=) the values. I used square brackets to enclose the values that are separated by commas. It goes: Name:= [values]
Third, call the map function which will generate a list of output values for a function given a list of input values.

```map(f, rightlist); [-0.68, -1.37, -1.9397, -1.993997, -1.99939997, -1.9999400000]```

```map(f, leftlist); [-2.57, -2.0597, -2.005997, -2.000006000, -2.0000000000]```

Finally, I interpret what I get. It looks like as I put values a little bit bigger than 1, but getting closer to 1 my output approaches -2.0. It looks like as I put values a little bit smaller than 1, but getting closer to 1 my output approaches -2.0. My final evaluation is that it appears that

\[
\lim_{x \to 1} f(x) = -2.0.
\]

One word of caution: this is one approach to finding a limit, there are cases in which this approach may lead you astray if you are not careful.

A final note about the map command: it is important that you put the name of the function and not the function itself. Remember, to Maple, the function itself is given by the expression \(f(x)\) and the name is just \(f\). Look at what happens if I type in the function:

```map(f(x), leftlist); [3 x(0.9)^2 - 5, 3 x(0.99)^2 - 5, 3 x(0.999)^2 - 5, 3 x(0.999999)^2 - 5, 3 x(0.999999999999)^2 - 5]```

**Method 2.** With the graphical method, a CAS is a huge help! All you need to do is recall how to use the "plot" command and choose the appropriate viewing rectangle. For example if I wanted to consider what a function was doing near \(x=1\), I would choose to plot the function for \(x\) values around \(x=1\).

```plot(f(x), x=0.9..1.1, title="f near x=1");```

Visually, it appears that as the \(x\) values approach 1, the output approaches -2. Hence I would claim that \(\lim_{x \to 1} f(x) = -2\). This method is good for estimating the value, but does not always produce a “rigorous” answer. Consider using this approach to calculate \(\lim_{x \to 0.5} f(x)\)
The visual approach is excellent for providing some intuition about what is going on. When an exact value is needed other methods must be employed. In these cases where an exact answer is needed one should employ methods 3 or 4 as well or go back to the exact definition of the limit.

**Method 3.** With the algebraic method, Maple can aid you with some of the calculations but it is primarily up to you to decide what to do. Consider the techniques you have been introduced to in class, such as rationalizing the denominator, and get out your paper and pencil. Some of the Maple commands that might come in handy are *factor, expand, simplify*. We will not cover this method in this lab, however you will be doing plenty of homework exercises that employ this important technique.

**Method 4.** With the symbolic method, Maple does the work for you if you type in the appropriate commands. There are times where the limit may be complicated enough and you must choose to use one of the other methods. At other times the limit may not exist and you must use your calculus skills to interpret what Maple is trying to tell you.

To find the limit \( \lim_{x \to 1} f(x) \) symbolically you first define your function and then use the limit command:

\[
> f := x \rightarrow x^2 + 3;
\]

\[
f := x^2 + 3
\]

\[
> \text{limit}(f(x), x=1);
\]

\[
4
\]

One-sided limits

Recall, there are also one-sided limits. Note, when you just type limit Maple will calculate the two sided limit when it exists. To find a right- or left-hand limit with the limit command, type "right" or "left" after the " x=a" part. Consider the following limits for \( f(x) = \frac{1}{x} \):

\[
\lim_{x \to 0^+} f(x) \quad \text{is typed in as} \quad > \text{limit}(1/x, x=0, \text{right});
\]

\[
\infty
\]

\[
\lim_{x \to 0^-} f(x) \quad \text{is} > \text{limit}(1/x, x=0);
\]

\[
\text{undefined}
\]
\[
\lim_{x \to 0^-} f(x) \text{ is typed in as } \texttt{limit}(1/x, x=0, \text{left});
\]

So the limit does not exist because the right and left handed limits differ. From the right of zero the function is growing without bound (to infinity) and from the left of zero the function is approaching really large negative values (negative infinity).

**Example**

As said above, you need to interpret Maple’s results. If we wish to compute \( \lim_{x \to 0} \sin(1/x) \)

\[
\texttt{limit}(\sin(1/x), x=0);
\]

it appears that Maple is telling us 2 answers, -1 and 1. What is going on? This is when employing some of the other methods can help.

\[
\texttt{plot}(\sin(1/x), x=-0.01..0.1);
\]

\[
\texttt{rightlist} := [.3, .001, .00004, .00006];
\]

\[
\texttt{map}(f, \texttt{rightlist});
\]

It looks like this function is oscillating back and forth between -1 and 1 the closer we get to zero. That is what Maple meant by -1..1. So the limit actually does not exist because the function does not approach a unique number as \( x \) approaches 0.

**Pretty print**

If you replace the "l" with "L" in the limit command Maple will just return to you what you told it to compute. This is helpful in determining if you typed in the function correctly and for typesetting your lab reports.

\[
\texttt{Limit(\sin(x)/x, x=0, \text{left});}
\]

\[
\lim_{x \to 0^-} \frac{\sin(x)}{x}
\]

\[
\texttt{Limit(\sin(x)/x, x=0, \text{left}):%=\text{value}%;}
\]
\[
\lim_{x \to 0} \frac{\sin(x)}{x} = 1
\]

**Exercises:**

**Part 1**

1. As discussed above, the tabular method can get you in the vicinity of the limit but can also lead you astray at times. We are going to examine \( \lim_{x \to 0} f(x) \) for \( f(x) = \frac{5^x - 1}{x} \) using the tabular method which is called method 1 in this lab.

   a. Use Maple’s map and sequence commands to fill out the following charts:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1/100000</th>
<th>1/1000000</th>
<th>1/10000000</th>
<th>( \frac{1}{10^0} )</th>
<th>( \frac{1}{10^1} )</th>
<th>( \frac{1}{10^2} )</th>
<th>( \frac{1}{10^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td>-1/100000</td>
<td>-1/1000000</td>
<td>-1/10000000</td>
<td>( \frac{-1}{10^0} )</td>
<td>( \frac{-1}{10^1} )</td>
<td>( \frac{-1}{10^2} )</td>
<td>( \frac{-1}{10^3} )</td>
</tr>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   Remember how to do this, first create a list of values (read up above how to do this). Then use Maple’s map command and fill in the chart. Finally, interpret your charts to give the value of \( \lim_{x \to 0} f(x) \) for \( f(x) = \frac{5^x - 1}{x} \).

   b. To make sure of your guess let’s get even closer to zero in both of the lists. Create a similar table of values with \( x = \frac{1}{10^i} \) and \( x = \frac{-1}{10^i} \) for \( i = 12, 13, 14 \) and \( 15 \).

   What does it look like the output is approaching? Note: this is your limit.

   c. Next examine it graphically by plotting the function in the appropriate viewing window. What does it look like the limit is? Does this verify or contradict your results from parts a and b?

   d. Later in the course you will learn techniques to deal with this limit so we will skip method 3 for now. Employ method 4 by using Maple’s limit command. What do you get?

   e. What is the limit? Why do you think this is?

**Part 2**

1. Not all of your limit investigations involve approaching the value 0. Explain how to adapt the sequence commands we gave in the lab to get a list of 12 values closer and closer to 5?
2. Create 2 lists with the sequence command. Name the first one “right” and have it contain the 5 values –2.001, -2.0001, -2.00001, -2.000001, -2.0000001. Name the second one “left” and have it contain the 4 values –1.99, -1.999, -1.9999, -1.99999. Show me the input you used and the output generated.

Part 3

Examine the following limits by methods 1, 2, 4 discussed above. For each method answer the following:

a) What did you type in?
b) What output did you get? If it is a graph provide a minimized picture.
c) What is your interpretation of your output? (i.e. what does it look like the limit is)

Finally, summarize your results and state your final answer to the limit and if the limit does not exist, explain why.

1. limit as x->10 in f(x)=1/(x-10)
2. limit as x-> \( \pi \) in \( f(x) = \sin\left(\frac{1}{x-\pi}\right) \)
3. limit as x->1 in \( \frac{\sin\left(\frac{1}{x}\right)}{x-1} \)
4. \( \lim_{y \to 6} (y - 2)^{1/3} \)
5. \( \lim_{r \to 1000} 210\left(\frac{11}{21}\right)^{r/10} + 85 \)

Part 4

Examine the following limits, use method 4.

1. Let \( f(x) = -x^2 \). In class we defined the expression \( \frac{f(x+h) - f(x)}{h} \) to be the average velocity over the time interval \((x, x+h)\) if \( f(x) \) was a distance function. We also saw that it was the slope of the secant line that connected the two points \((x, f(x))\) and \((x+h, f(x+h))\). If we wanted the instantaneous velocity at time \( x \), we would find \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \). Do this using method 4 of this lab. Use your result to find the instantaneous velocity at time 4 seconds.
2. Let \( g(x) = x^3 \), find \( \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \) and \( \lim_{x \to a} \frac{g(x) - g(a)}{x-a} \). What do you notice? If \( g(x) \) is a distance function, what do these 2 limits calculate?

**Part 5**
Use method 2 and method 4 to find the following limits.

1. The limit as \( x \) approaches 2 in \( f(x) = \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} \).

2. \( \lim_{t \to -3} e^{\frac{1}{t+3}} \)

3. \( \lim_{t \to 16} \frac{\sqrt{t} - 4}{t - 16} \)