Limits at Infinity and Other Asymptotes

Introduction and Goals

This lab will get into some more advanced techniques, as well as revisit some old ones, for graphing and analyzing functions. Its main purpose is to show you a little more of the power of Maple and how to make certain manipulations easier.

Before You Start

Before you start you may want to review the Maple commands in the previous labs.

Textbook Correspondence

Stewart 5th edition: 2.6.

Maple Commands and Packages Used

Commands: plot, limit, rem, quo.

Maple Commands

Exploring Asymptotes A Bit Further:

In class we have looked at vertical and horizontal asymptotes. Recall that vertical asymptotes occurred when a limit was infinite and horizontal asymptotes were when limits at infinity existed. In Maple, a limit at infinity is done like any other limit except that instead of placing \( x = 2 \) in the limit we place \( x = \infty \) or \( x = -\infty \) in the limit. For example,

\[
> \text{limit}(\arctan(x), x=\infty) ; \quad \frac{\pi}{2}
\]

\[
> \text{limit}(\arctan(x), x=-\infty) ; \quad -\frac{\pi}{2}
\]

\[
> \text{limit}((2*x^2-3*x+4)/(x^2-5*x+9), x=\text{infinity}) ; \quad 2
\]

\[
> \text{limit}(x^2, x=\text{infinity}) ; \quad \infty
\]
In general, an asymptote is a line or curve that a function approaches. For example, the function
\[
\frac{4x^3 - 3x^2 + 17x - 5}{x^2 + x + 1}
\]
gets close to
\[
4x - 7
\]
as the value of \(x\) increases or decreases. This can easily be seen in the graph,

\[
> \text{plot}([ (4x^3-3x^2+17x-5)/(x^2+x+1) , 4\cdot x-7], x=-10..10);
\]

We will only consider asymptotes to rational functions. In our above example note that
\[
\frac{4x^3 - 3x^2 + 17x - 5}{x^2 + x + 1} = 4x - 7 + \frac{2 + 20x}{x^2 + x + 1}
\]
The rational function
\[
\frac{2 + 20x}{x^2 + x + 1}
\]
goes to 0 as the value of \(x\) increases or decreases. Hence as the value of \(x\) increases or decreases
\[
\frac{4x^3 - 3x^2 + 17x - 5}{x^2 + x + 1}
\]
gets close to
\[
4x - 7
\]
We call the \(4x - 7\) an asymptotic function. For rational functions finding the asymptotic function is easy, note that
\[
4x - 7
\]
is simply the quotient when you do the long division of

\[
\frac{4x^3 - 3x^2 + 17x - 5}{x^2 + x + 1}
\]

In Maple the quotient is found using the quo command. For example,

\[
> \text{quo}(4x^3-3x^2+17x-5, x^2+x+1, x);
\]

\[
4x - 7
\]

As another example,

\[
> f:=x\rightarrow (x^4-10x^3+3x^2-7x+9)/(2x^2+9x-5);
\]

\[
f := x \mapsto \frac{x^4 - 10x^3 + 3x^2 - 7x + 9}{2x^2 + 9x - 5}
\]

\[
> \text{quo}(x^4-10x^3+3x^2-7x+9, 2x^2+9x-5, x);
\]

\[
\frac{1}{2}x^2 - \frac{29}{4}x + \frac{283}{8}
\]

\[
> \text{plot}([f(x), 1/2*x^2-29/4*x+283/8], x=-20..30, y=-400..400, \text{discont}=\text{true});
\]

Notice the use of the option discont=true. Recall that this command removes the graphing of vertical asymptotes if it can.
Exercises:

1. For each of the following functions find all vertical asymptotes, horizontal asymptotes and/or asymptotic functions. Graph the function along with its asymptotes, use ranges for x and y that display the fact that the function is approaching its asymptote(s).

\[
\frac{4x^3 + 8x^2 - 4x + 26}{2x^2 + x - 2} \quad \frac{x^2 - 25x + 2}{6x^2 + x - 2} \\
\frac{x^4 + 5x^2 - 7x + 2}{2x^2 + x - 2} \quad \frac{5x + 2}{8x^2 + 6x - 2} \\
\frac{6x^6 + x^5 - 7x^4 + 4x^3 + 7x^2 - 25x + 2}{x^2 + x - 2}
\]

2. This exercise is simply an analysis of the function,

\[
x \left( \frac{13229}{5000} \right) \\
x \left( \sqrt[7]{x} \right) - x + 2
\]

a. Define this function as \( f(x) \).
b. Graph this function on the intervals \([0, 100], [0, 1000], [0, 10000], [0, \infty)\). Does the function have a horizontal asymptote? If so, where? What does this say about \( \lim_{x \to \infty} \frac{x}{\left( \sqrt[7]{x} \right) - x + 2} \)?

c. Use numeric techniques, with very large values of \( x \), to determine

\[
\lim_{x \to \infty} \frac{x}{\left( \sqrt[7]{x} \right) - x + 2}
\]
d. Use Maple’s limit command to find

\[
\lim_{x \to \infty} \frac{x}{\left( \sqrt[7]{x} \right) - x + 2}
\]
e. Use the techniques discussed in class to determine

\[
\lim_{x \to \infty} \frac{x}{\left( \sqrt[7]{x} \right) - x + 2}
\]
f. Take all of the above information and discuss your findings. I am expecting several paragraphs of thoughtful comments and observations.