Lagrange Multipliers

**Introduction and Goals:**

The goal of this lab is to become more familiar with the process and workings of Lagrange multipliers. This lab is designed more to help you understand the reason why the Lagrange multiplier method works than it is to use Maple to solve more difficult problems.

**Before You Start:**

Make sure that you read and understand the mathematics from the corresponding sections in your textbook.

**Textbook Correspondence:**


**Maple Commands and Packages Used:**

Packages: plots, VectorCalculus.
Commands: plot3d, solve, evalf, Del, implicitplot, implicitplot3d, contourplot, display, spacecurve.

**History & Biographies:**

**Maple Commands:**

As you know from the problem sets in your text, Lagrange multipliers can be rather tricky to solve. You usually have between three and five variables to solve for and at times you need to do some fancy footwork to obtain all of the possibilities. This can be very hard for us lowly humans to do but it can be even more difficult for a computer. The fancy footwork that we sometimes need to do often involves manipulating the equations in a non-algorithmic or even intuitive way. Intuition is one thing we have not been able to program into a computer, yet. So sometimes Maple will be able to solve the equations we get in the Lagrange multiplier process, sometimes it will get a few solutions and other times it will fail miserably in obtaining the solutions. As a result, we can use Maple to help solve these equations and we can use the graphics that Maple has to get a better understanding of how and why the process works. This lab is devoted more to using Maple to help visualize the Lagrange multiplier process than it is to using Maple to
solve Lagrange multiplier problems. Before we get started we need to load two packages into Maple, plots and VectorCalculus.

> \texttt{with(plots)}:
Warning, the name changecoords has been redefined

> \texttt{with(VectorCalculus)}:
Warning, the assigned names \texttt{<,> and <|> now have a global binding}
Warning, these protected names have been redefined and unprotected: \texttt{*}, \texttt{+, ., Vector, diff, int, limit, series}

Let’s start with a relatively simple example. Maximize the function \( f(x, y) = 4x + 6y \) under the constraint \( x^2 + y^2 = 13 \). If we approach this from a purely algorithmic point of view we define the function to be maximized and the constraint.

> \texttt{f:=(x,y)\rightarrow 4*x+6*y;}
\( f := (x, y) \to 4x + 6y \)

> \texttt{g:=(x,y)\rightarrow x^2+y^2;}
\( g := (x, y) \to x^2 + y^2 \)

We then find their gradient vectors.

> \texttt{df:=Del(f(x,y),[x,y]);}
\( df := \nabla f(x, y) = 4\mathbf{e}_x + 6\mathbf{e}_y \)

> \texttt{dg:=Del(g(x,y),[x,y]);}
\( dg := \nabla g(x, y) = 2x\mathbf{e}_x + 2y\mathbf{e}_y \)

Solve the equations \( \nabla f = \lambda \nabla g \) and \( x^2 + y^2 = 13 \), simultaneously to get the possibilities for \( x, y \) and \( \lambda \).

> \texttt{solve\{df[1]=lambda*dg[1], df[2]=lambda*dg[2], g(x,y)=13\},
\{x,y,lambda\};}
\{ \lambda = 1, x = 2, y = 3 \}, \{ y = -3, x = -2, \lambda = -1 \}

Finally, we take each of the solutions and evaluate them in the function \( f \).

> \texttt{f(2,3);}
26

> \texttt{f(-2,-3);}
-26
Hence the maximum is 26 at the point \((2,3)\) and the minimum is \(-26\) at the point \((-2,-3)\). Our conclusion relies heavily on the assumption that the solve command produced all of the possible solutions to the equations. In this example it has but in general this is a very big assumption. We will see an example later where the solve command gives us only a few of the possible answers. Before we move onto another example lets look at this example a little closer. As you know from your reading the reason that we solve the equation \(\nabla f = \lambda \nabla g\) along with the constraint \(x^2 + y^2 = 13\), is to force the constraint to be tangent to a level curve of the function we are maximizing or minimizing. We will take two views of this situation. First we will graph the constraint with the contour map of the surface.

```maple
display(implicitplot(g(x,y)=13,x=-4..4,y=-4..4), contourplot(f(x,y),x=-4..4,y=-4..4,contours=16));
```

Notice that the constraint, the circle, is tangent to a level curve of the function at two points, \((2,3)\) and \((-2,-3)\). These are precisely the points that the solve command gave us above. Now we will look at this same situation by projecting the constraint on the surface. To do this we need to graph the surface and a space curve. The only difficulty is that to graph the space curve we need to parameterize the constraint equation. In some cases this is difficult but here it is a snap since the constraint is a circle centered at the origin with radius \(\sqrt{13}\). So the parameterization of this curve is

\[
x = \sqrt{13} \cos(\theta) \\
y = \sqrt{13} \sin(\theta)
\]

To display both the curve and surface we use the display command with the plot3d and spacecurve commands.

```maple
display(plot3d(f(x,y),x=-4..4,y=-4..4), spacecurve([sqrt(13)*cos(t),sqrt(13)*sin(t), f(sqrt(13)*cos(t),sqrt(13)*sin(t))],t=0..2*Pi,color=black, thickness=3),axes=boxed);
```
Think of yourself as being in the mountains hiking on a trail that follows the constraint curve and the surface is the side of the mountain. As you hike the trail you will either gain altitude or lose altitude. When you are approaching the point of highest altitude, the maximum, you start walking in a direction that does not gain or lose any altitude. If you were still gaining altitude you would not be close to the maximum. If you are walking in a direction that is not gaining or losing altitude then you would be following a contour, since contour lines are by definition line on a surface that do not gain or lose altitude. If you are waking in a direction that follows a contour at the maximum altitude then your path must be tangent to a contour at that point. The same holds for minimums. If you adjust the view of the above plot to that you are looking down the $z$-axis and then press the button that displays the contour on the surface you get an image similar to the one that we generated above.
Let’s look at another example. Find the maximum and minimum of \( f(x, y) = x^2 + y^2 \) under the constraint \( x^4 + y^4 = 1 \). First define the functions for the surface and constraint.

\[
\begin{align*}
  f &:=(x, y) \rightarrow x^2 + y^2; \\
  g &:=(x, y) \rightarrow x^4 + y^4;
\end{align*}
\]

Find their gradients

\[
\begin{align*}
  df &= \text{Del}(f(x, y), [x, y]); \\
  dg &= \text{Del}(g(x, y), [x, y]);
\end{align*}
\]

and solve the system of equations.

\[
\begin{align*}
  \text{solve}\{df[1]=\lambda*dg[1], df[2]=\lambda*dg[2], g(x, y)=1\}, \\
  \{x, y, \lambda\};
\end{align*}
\]

Note that we get 8 solutions, two of which are imaginary. So we have six real solutions in all. As we will see later, we are missing a couple. Let’s take a look at the situation graphically. The constraint looks like the following,

\[
\text{implicitplot}(g(x, y)=1, x=-1..1, y=-1..1);
\]
The surface of interest is as below.

\[ \text{plot3d}(f(x,y), x=-1.5..1.5, y=-1.5..1.5, axes=boxed); \]

To view the constraint on the surface we can either parameterize the curve, which we will do later, or we can turn the constraint curve into a cylinder and examine the intersection of the surface and cylinder and surface.

\[ \text{display(plot3d}(f(x,y), x=-1.5..1.5, y=-1.5..1.5, axes=boxed), \text{implicitplot3d}(g(x,y)=1, x=-1..1, y=-1..1, z=0..4)); \]
As we can see from the graph there seems to be maxima at the “corners” of the constraint and minima at the “edges”. If we look straight down the \( z \)-axis and press the contour button we see that the constraint and the contours are tangent at the corners and the edges.

This gives a total of 8 real solutions, so we are missing two from our solve command above. First note that the decimal number 0.8408964155 in the solve output is actually \( \frac{1}{\sqrt[4]{2}} \).

\[
> \text{evalf}(1/2^{1/4});
\]

\[
0.8408964155
\]
So our solve command produced \((0, \pm 1), (\pm 1, 0)\) and \((\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})\). The missing points are \((\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{3\sqrt{2}})\).

\[
> f(1,0);
\]
1

\[
> f(-1,0);
\]
1

\[
> f(0,1);
\]
1

\[
> f(0,-1);
\]
1

\[
> f(1/2^(1/4),1/2^(1/4));
\]
\(\sqrt{2}\)

\[
> f(-1/2^(1/4),1/2^(1/4));
\]
\(\sqrt{2}\)

\[
> f(1/2^(1/4),-1/2^(1/4));
\]
\(\sqrt{2}\)

\[
> f(-1/2^(1/4),-1/2^(1/4));
\]
\(\sqrt{2}\)

So our maximum is \(\sqrt{2}\) at all four corners and our minimum is 1 at each of the four edges. If we wanted to create an image similar to the one from the last example, that is, with the constraint curve plotted on the surface, we need to first parameterize the restraint curve. Since the constraint is \(x^4 + y^4 = 1\) the obvious parameterization is

\[
x = \sqrt{\cos(\theta)}
\]
\[
y = \sqrt{\sin(\theta)}
\]

Unfortunately, these do not give the entire curve, they simply produce one corner. For example,

\[
> plot([sqrt(cos(t)), sqrt(sin(t)), t=0..2*Pi]);
\]
We can however use four parameterizations to construct the entire constraint. For example,

\[
\begin{align*}
p_1 & := \text{plot}\left(\left[\sqrt{\cos(t)}, \sqrt{\sin(t)}, t = 0..2\pi\right]\right); \\
p_2 & := \text{plot}\left(\left[-\sqrt{\cos(t)}, \sqrt{\sin(t)}, t = 0..2\pi\right]\right); \\
p_3 & := \text{plot}\left(\left[-\sqrt{\cos(t)}, -\sqrt{\sin(t)}, t = 0..2\pi\right]\right); \\
p_4 & := \text{plot}\left(\left[\sqrt{\cos(t)}, -\sqrt{\sin(t)}, t = 0..2\pi\right]\right); \\
display(p1,p2,p3,p4);
\end{align*}
\]

To project this onto the surface we simply create space curves from each component where the \(z\) value is on the surface.

\[
\begin{align*}
p_5 & := \text{plot3d}(x^2+y^2, x = -1.5..1.5, y = -1.5..1.5);
\end{align*}
\]
One further thing we can do for a better visual effect is to also place the constraint curve on the \(xy\)-plane. Again we can do this with space curves.

```maple
> p11:=spacecurve([sqrt(cos(t)),sqrt(sin(t)),0],t=0..2*Pi, color=black,numpoints=500,thickness=3):
> p12:=spacecurve([-sqrt(cos(t)),sqrt(sin(t)),0],t=0..2*Pi, color=black,numpoints=500,thickness=3):
> p13:=spacecurve([sqrt(cos(t)),-sqrt(sin(t)),0],t=0..2*Pi, color=black,numpoints=500,thickness=3):
> p14:=spacecurve([-sqrt(cos(t)),-sqrt(sin(t)),0],t=0..2*Pi, color=black,numpoints=500,thickness=3):
> display(p5,p6,p7,p8,p9,p11,p12,p13,p14,axes=boxed);
```
Let’s move up one dimension now. Say we wanted to find the maxima and minima of the function \( f(x, y) = x^2 y^2 z^2 \) under the constraint of the unit sphere, that is \( x^2 + y^2 + z^2 = 1 \). Unfortunately, we can no longer graph the surface since it is a function of three variables. As with the functions of two variables we define the functions and take their gradients.

\[
\begin{align*}
  f(x, y, z) &= x^2 y^2 z^2 \\
  g(x, y, z) &= x^2 + y^2 + z^2
\end{align*}
\]

Solving the system of equations gives,

\[
\begin{align*}
  x^2 \ddot{x} + 2 x y \ddot{y} + 2 x z \ddot{z} &= 0 \\
  y^2 \ddot{y} + 2 y z \ddot{z} &= 0 \\
  z^2 \ddot{z} &= 0
\end{align*}
\]

\[
\begin{align*}
  \lambda &= \frac{\ddot{x}}{x^2} = \frac{\ddot{y}}{y^2} = \frac{\ddot{z}}{z^2}
\end{align*}
\]
\{x = 0, \lambda = 0, y = \text{RootOf}(_Z^2 - 1 + z^2), z = z\}, \\
\{y = 0, \lambda = 0, z = z, x = \text{RootOf}(_Z^2 - 1 + z^2)\}, \\
\{z = 0, \lambda = 0, y = y, x = \text{RootOf}(_Z^2 - 1 + y^2)\}, \\
\{z = \text{RootOf}(3 _Z^2 - 1), x = \text{RootOf}(3 _Z^2 - 1), y = \text{RootOf}(3 _Z^2 - 1), \lambda = \frac{1}{9}\}, \\
\{z = \text{RootOf}(3 _Z^2 - 1), y = -\text{RootOf}(3 _Z^2 - 1), x = -\text{RootOf}(3 _Z^2 - 1), \lambda = \frac{1}{9}\}, \\
\{z = \text{RootOf}(3 _Z^2 - 1), y = -\text{RootOf}(3 _Z^2 - 1), x = \text{RootOf}(3 _Z^2 - 1), \lambda = \frac{1}{9}\}, \\
\{z = \text{RootOf}(3 _Z^2 - 1), x = -\text{RootOf}(3 _Z^2 - 1), y = \text{RootOf}(3 _Z^2 - 1), \lambda = \frac{1}{9}\} \\
> \text{evalf}(%)%; \\
\{y = \text{RootOf}(_Z^2 - 1 + z^2), z = z, \lambda = 0, , x = 0.\}, \\
\{y = 0., z = z, x = \text{RootOf}(_Z^2 - 1 + z^2), \lambda = 0.\}, \\
\{z = 0., y = y, x = \text{RootOf}(_Z^2 - 1 + y^2), \lambda = 0.\}, \\
\{y = 0.5773502693, \lambda = 0.1111111111, z = 0.5773502693, x = 0.5773502693\}, \\
\{\lambda = 0.1111111111, z = 0.5773502693, y = -0.5773502693, x = -0.5773502693\}, \\
\{\lambda = 0.1111111111, z = 0.5773502693, y = -0.5773502693, x = 0.5773502693\}, \\
\{y = 0.5773502693, \lambda = 0.1111111111, z = 0.5773502693, x = -0.5773502693\} \\
> \text{evalf}(1/\sqrt(3)); \\
0.5773502693

As you can see, there seems to be 4 real solutions, \((\pm \frac{1}{\sqrt(3)}, \frac{1}{\sqrt(3)}, \frac{1}{\sqrt(3)})\) and \((\pm \frac{1}{\sqrt(3)}, -\frac{1}{\sqrt(3)}, \frac{1}{\sqrt(3)})\), each occurring when \(\lambda = \frac{1}{9}\). Maple also outputs some unreadable solutions for \(\lambda = 0\). Solve the system by hand, you should obtain four more solutions, \((\pm \frac{1}{\sqrt(3)}, \frac{1}{\sqrt(3)}, -\frac{1}{\sqrt(3)})\) and \((\pm \frac{1}{\sqrt(3)}, -\frac{1}{\sqrt(3)}, -\frac{1}{\sqrt(3)})\), for \(\lambda = \frac{1}{9}\). You should also note that when \(\lambda = 0\) we may have any one or two (but not three) coordinates equaling 0. This produces the following set of possibilities: \((0, y, z)\), \((x, 0, z)\), \((x, y, 0)\) where the x, y and z must satisfy the constraint. Again Maple does not produce all of the possibilities. From this it is easy to see that the minimum is 0 and the maximum is \(\frac{1}{27}\). Although we cannot graph the function \(f\) we can graph its level surfaces along with the constraint. As we saw in the two-dimensional case the constraint should be tangent to the level surface at the maximum. To graph the level surface of \(f\) at the maximum we graph the implicit equation \(f(x, y, z) = \frac{1}{27}\).

\(> \text{p20:=implicitplot3d}(f(x, y, z)=1/27, x=-1..1, y=-1..1, z=-1..1):\)
\(> \text{p21:=implicitplot3d}(g(x, y, z)=1, x=-1..1, y=-1..1, z=-1..1):\)
\(> \text{display(p20,p21);}\)
Notice that the level surface and the constraint are tangent at eight points on the surface, corresponding to the points \((\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})\). Now consider the following image, obtained from graphing the constraint along with the level surface \(f(x, y, z) = \frac{1}{2}\).

\[
> p22:=\text{implicitplot3d}(f(x, y, z) = 1/2, x=-2..2, y=-2..2, z=-2..2, \text{grid}=[20, 20, 20]);
> \text{display}(p22,p21);
\]
Notice that there are no points of intersection between the level surface and the constraint. Hence, the function never attains the value of $\frac{1}{2}$ on the constraint and thus $\frac{1}{2}$ can not be a maximum or a minimum. Now consider the level surface $f(x, y, z) = \frac{1}{1000}$ graphed along with the constraint.

```
p23:=implicitplot3d(f(x, y, z)=1/1000, x=-2..2, y=-2..2, z=-2..2, grid=[20,20,20]):
display(p23,p21);
```

Since there is an intersection between the level surface and the constraint the function does attain the value $1/1000$ over the constraint. Hence the graph is telling us that the maximum of the function over the constraint is at least $1/1000$ and it is probably higher. Thinking of this dynamically, if we consider level surfaces starting at 0 and increasing, as long as the constraint intersects the level surface there are points on the constraint in which the function attains the surface value. We keep increasing the value until the moment where the constraint and level surface no longer intersect. At this moment, the level value is the maximum of the function on the constraint, and consequently the level surface and constraint are tangent.

Unfortunately, the graphical image of the minimum of the function over the constraint is not as satisfying. Recall that the minimum of this function over the constraint is 0. This is, of course, the absolute minimum of the function as well. If we were to graph
\( f(x, y, z) = 0 \) on Maple we would get an empty plot. The actual image of this level surface would be the three coordinate planes. There is, of course, an intersection between the constraint and the three coordinate planes. Hence the function does attain its absolute minimum at an infinite number of points on the constraint even though the level surface is not tangent to the constraint.

Exercises:

1. The following sequence of exercises are to find and analyze the maximum and minimum of the function \( f(x, y) = \sin(x - y) \) on the constraint \( x^2 + y^2 = 1 \).
   a. Use Maple to define the functions, find their gradients and solve the system of equations. How many real solutions did you get for the system?
   b. Using the real solutions to the system determine the absolute maximum and minimum of the function over the constraint.
   c. Create the following graphs:
      i. The surface along with the constraint, viewed on the surface.
      ii. The constraint with a contour plot of the surface.
      iii. The height of the surface on the constraint curve as a function of the curve parameter \( t \) in the range \([0, 2\pi]\).
         From the graphs, how many real solutions should there be to the system? If there are any missing, find them.
   d. Using the new set of real solutions to the system determine the absolute maximum and minimum of the function over the constraint.

2. The following sequence of exercises are to find and analyze the maximum and minimum of the function \( f(x, y) = \sin(x - y) \) on the constraint \( x^2 + y^2 = 2 \).
   a. Use Maple to define the functions, find their gradients and solve the system of equations. How many real solutions did you get for the system?
b. Using the real solutions to the system determine the absolute maximum and minimum of the function over the constraint.

c. Create the following graphs:
   i. The surface along with the constraint, viewed on the surface.
   ii. The constraint with a contour plot of the surface.
   iii. The height of the surface on the constraint curve as a function of the curve parameter \( t \) in the range \([0,2\pi]\).

From the graphs, how many real solutions should there be to the system? If there are any missing, find them.

d. Using the new set of real solutions to the system determine the absolute maximum and minimum of the function over the constraint.

e. What differences are there in your analysis of this problem as opposed to the previous exercise? Does there appear to be a discrepancy in the geometry behind the Lagrange multiplier method? Why or why not?

3. The following sequence of exercises are to find and analyze the maximum and minimum of the function \( f(x, y, z) = x^3 + 3x^2z + 2y^2z \) on the unit sphere centered at the origin.

   a. Use Maple to define the functions, find their gradients and solve the system of equations. How many real solutions did you get for the system?

   b. Solve the system of equations by hand. Did you get any new points of interest? Did you get any new values for the multiplier?

   c. Using the solutions to the system determine the absolute maximum and minimum of the function over the constraint.

   d. Graph the level surface at the maximum along with the constraint. Does your graph correspond with the solutions you derived?

   e. Graph the level surface at the minimum along with the constraint. Does your graph correspond with the solutions you derived?

4. The following sequence of exercises are to find and analyze the maximum and minimum of the function \( f(x, y, z) = x^3 + y^3 - z^3 \) on the unit sphere centered at the origin.

   a. Use Maple to define the functions, find their gradients and solve the system of equations. How many real solutions did you get for the system?

   b. Solve the system of equations by hand. Did you get any new points of interest? Did you get any new values for the multiplier?

   c. Using the solutions to the system determine the absolute maximum and minimum of the function over the constraint.

   d. Graph the level surface at the maximum along with the constraint. Does your graph correspond with the solutions you derived?

   e. Graph the level surface at the minimum along with the constraint. Does your graph correspond with the solutions you derived?

5. The following sequence of exercises are to find and analyze the maximum and minimum of the function \( f(x, y, z) = x^3 + y^3 - z^3 \) on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 1 \).

   a. Use Maple to define the functions, find their gradients and solve the system of equations. How many real solutions did you get for the system?
b. Solve the system of equations by hand. Did you get any new points of interest? Did you get any new values for the multiplier?
c. Using the solutions to the system determine the absolute maximum and minimum of the function over the constraint.
d. Graph the level surface at the maximum along with the constraint. Does your graph correspond with the solutions you derived?
e. Graph the level surface at the minimum along with the constraint. Does your graph correspond with the solutions you derived?
6. Use Maple to help you find the absolute maximum and minimum of the function \( f(x, y, z) = x^3 + y^3 - z^3 \) on the plane \( x + y + z = 0 \). Is there a major difference between this exercise and the ones above? If so, how?