A View Into Four and More Dimensions

In mathematics, especially in Calculus III and Linear Algebra, we frequently do calculations and find objects that are in four or more dimensions. These calculations are done without the aide of a visual interpretation of the situation since drawing four-dimensional objects on paper or a computer is a bit difficult. This project is designed to give you a small interpretation of what four-dimensions look like and what some of our three-dimensional notions convert to in four-dimensions. To start, we will consider the jump from two to three dimensions, not as three-dimensional beings as we are but as two-dimensional beings.

Imagine that you and two of your friends are two-dimensional beings. You have absolutely no knowledge of three dimensions. For example say that you are a circle and your friends are a square and a right triangle, as pictured below.

Think about how the three of you would move around in your two-dimensional world. What would your friends look like from your point of view? Could you tell your two friends apart, from your point of view? If so how, and if not what would you need to do in order to distinguish between your two friends? What situations are similar to these in our three dimensional world? How can we move around in our world? Are there ever times when you have difficulty distinguishing between two individuals? What do you do in order to distinguish between the people? What if your two-dimensional friends had something printed on the backs of their tee shirts? Could you read it from your current position? Now jump out of the two-dimensional world for a minute. Could you read the back of the shirt from your current three-dimensional world? If a four-dimensional being were looking down on you and your friends as you are looking down on these figures, could the four-dimensional being read the backs of the three-dimensional tee shirts?

Let’s concentrate on how a two-dimensional being would perceive you, a three-dimensional being, if you were to enter their world. As an experiment, take a glass of water and consider the two-dimensional being’s world as the surface, and only the surface, of the water. Slowly stick your index finger into the water. How would a two-dimensional being on the surface of the water perceive the events that just took place? How would a two-dimensional being on the surface of the water view your index finger? Would their perception of your index finger be anything like what you finger really is?
you were able to communicate with these two-dimensional beings, that have only ever experienced forward, back, left and right, how would you explain up and down?

Go back to the glass of water. Place two fingers in the water. How would a two-dimensional being on the surface of the water view your fingers? How many two-dimensional beings would he or she perceive? Spread your fingers apart and then move them together. How would a two-dimensional being on the surface of the water perceive the events that just took place? Now move your hand further into the glass of water, up to your top knuckle. How would a two-dimensional being on the surface of the water perceive the events that just took place? Imagine that the being on the surface of the water is a circle with a diameter of 2 inches. Could you stick your finger inside the being? How would a two-dimensional being on the surface of the water perceive the events that just took place?

Using your observations from above, consider what a four-dimensional being might be able to do in our three-dimensional world. Draw analogies from all of the above two and three-dimensional situation to the three and four-dimensional situation.

Now let’s look at this situation with the help of Maple. Say we are two dimensional again. What would happen if the three dimensional surface \( z = -x^4 - y^4 + 10x^2 + 10y^2 + 1 \) entered our two-dimensional world? The surface is below.

So say our world is represented by the plane in the figure below. What would we see as a two-dimensional person in the center of the plane as this surface moved through our world?
To get an idea of this start up Maple and execute the with(plots): command,

> with(plots):

Now execute the following command,

> animate3d({-x^4-y^4+10*x^2+10*y^2+1, z}, x=-3..3, y=-3..3, z=0..60, frames=20);

This will create the following image.

When you click on the image in Maple you will see the following toolbar.

This is the animation toolbar. To move through the frames of the animation simply click on the button. You can also change the direction of the animation by clicking on the buttons. You will also see the following on the right side of the toolbar.

This is a toolbar selector. You can get to the other toolbars by selecting the up and down arrows. Click on the button until the plane is at the top of the surface and then click this key to view the plane dropping through the surface. We are really thinking about the surface coming up through the plane (our two-dimensional world). What would we see as the surface moved through our world? If we were sitting in the middle of the plane, how would we perceive these events? Would we have any concerns as the surface moved through our world?

Now let's move up one dimension. Consider the hyper-surface \( w^2 + x^2 + y^2 + z^2 = 100 \). This surface is a three-dimensional surface that exists in four-dimensional space. Just as we let a plane drop through the above surface we will let a hyper-plane \( w = k \) (a constant) drop through this surface.
In Maple, execute the commands,

```maple
> w:=-10;
> implicitplot3d(w^2+x^2+y^2+z^2=100,x=-11..11,y=-11..11,z=-11..11,grid=[15,15,15]);
```

Execute these two commands incrementing the value of \( w \) each time until the value of \( w \) reaches 10. That is, use the values –10, –9, –8, … 8, 9, 10 for \( w \). Note what the image looks like at each value of \( w \). You have just witnessed a four-dimensional object pass through three-dimensional space. In other words, you actually saw the slices of the object as the object moved through our world. What did you see? How did you perceive the four-dimensional object?

Now consider the hyper-surface \( w=x^2-y^2+z^2 \). Execute the following two commands incrementing the value of \( w \) each time. Use the values –300, –200, –100, –50, –10, 0, 10, 50, 100, and 300 for \( w \). Note what the image looks like at each value of \( w \). You have just witnessed another four-dimensional object pass through three-dimensional space. What did you see? How did you perceive the four-dimensional object?

```maple
> w:=300;
> implicitplot3d(x^2-y^2+z^2=w,x=-20..20,y=-20..20,z=-20..20,grid=[15,15,15]);
```

Now let’s get a little more mathematical. We tend to deal with higher dimensions by taking what we know in lower dimensions and extending it. For example, how would we construct a four-dimensional cube? Easy, think about how we would construct a three-dimensional cube and do the same thing. Okay, how do we construct a three-dimensional cube? Easy, think about how we would construct a two-dimensional cube and do the same thing. Okay, how do we construct a two-dimensional cube? Easy, think about how we would construct a one-dimensional cube and do the same thing. Okay, how do we construct a one-dimensional cube? Easy, think about how we would construct a zero-dimensional cube and do the same thing. This is getting a bit out of hand. What is a zero-dimensional cube anyway? Well, the only zero-dimensional object I know about is a point. So I suppose that a zero-dimensional cube is a point. Fine. So what is a one-dimensional cube and how do I use zero-dimensional cubes to construct it? Well, the only one-dimensional objects I know are intervals of a line, so I suppose that a one-dimensional cube is a line segment. Okay, then constructing a one-dimensional cube from two zero-dimensional cubes is easy. Take two zero-dimensional cubes and connect them, like below.

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O-------------------O
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Well that is done. So how do I construct a two-dimensional cube? Well I suppose that a two-dimensional cube is a square. Easy, take two one-dimensional cubes and connect them. So start with,

![Diagram of a two-dimensional cube](image1)

and construct,

![Diagram of a two-dimensional cube](image2)

Now how do we construct a three dimensional cube? Take two two-dimensional cubes and connect them. So start with,

![Diagram of a three-dimensional cube](image3)

and construct

![Diagram of a three-dimensional cube](image4)
One last step, the four-dimensional cube. Just take two three-dimensional cubes and connect them. So start with,

and construct
The important thing to notice here is that we constructed a series of objects in exactly the same way, moving from dimension zero to dimension four. Now that we have taken a quick look into four dimensions let’s work with it.

1. A straight line in the plane that intersects the positive $x$ and $y$-axes will create a triangle between itself and the positive $x$ and $y$-axes. Derive a formula for the area of this triangle given that it intersects the $x$-axis at $a$ and the $y$-axis at $b$.

![Diagram of a triangle formed by a line intersecting the x and y axes.]

This may seem a bit on the trivial side, and it is, but you may want to derive the formula in a slightly different way. To get the area, use an integral in $x$ over the interval $[0, a]$ on the $x$-axis. This process may be useful later.

2. Consider the hyperbola $xy = k$ where $k$ is a constant. We will look at this function in only the first quadrant. Take a tangent line to the curve at any positive $x$ value, call it $x_0$. This tangent line will intersect the positive $x$ and $y$-axes. Using the formula derived in the previous exercise, determine the formula for the area of the triangle bounded by the tangent line and the positive $x$ and $y$-axes. Is there anything interesting about the formula you derived? If so, what?
3. A plane that intersects the positive $x$, $y$ and $z$-axes will create a tetrahedron with the positive $x$, $y$ and $z$-axes in the first octant. Derive a formula for the volume of this tetrahedron given that it intersects the $x$-axis at $a$, the $y$-axis at $b$ and the $z$-axis at $c$.

Again there are easier ways to do this but you may want to derive the formula by taking an integral in $x$ over the interval $[0, a]$ on the $x$-axis of the cross sectional areas shown below. Again, this process may be useful later.
4. Consider the hyperboloid \( xyz = k \) where \( k \) is a constant. We will look at this function in only the first octant. Take a tangent plane to the surface at any point \((x_0; y_0; z_0)\) where both \(x_0\) and \(y_0\) are positive. This tangent plane will intersect the positive \(x\), \(y\) and \(z\)-axes. Using the formula for the volume of the tetrahedron you derived above, determine the formula for the volume of the tetrahedron bounded by the tangent plane and the positive \(x\), \(y\) and \(z\)-axes. Is there anything interesting about the formula you derived? If so, what?

5. A hyper-plane that intersects the positive \(x\), \(y\) and \(z\)-axes will create a hyper-tetrahedron with the positive \(x\), \(y\), \(z\) and \(w\)-axes in the first "16-ant". Derive a formula for the hyper-volume of this hyper-tetrahedron given that it intersects the \(x\)-axis at \(a\), the \(y\)-axis at \(b\), the \(z\)-axis at \(c\) and the \(w\)-axis at \(d\). Use similar methods to those above. Do the same for five and six-dimensions, finally determine a formula for the hyper-volume of the hyper-tetrahedron in \(n\) dimensions.

6. Consider the hyperboloid \( xyzw = k \) where \( k \) is a constant. Take a tangent hyper-plane to the hyper-surface at any point \((x_0; y_0; z_0; w_0)\) where \(x_0\), \(y_0\) and \(z_0\) are all positive. This tangent hyper-plane will intersect the positive \(x\), \(y\), \(z\) and \(w\)-axes.
Using the formula for the volume of the hyper-tetrahedron you derived above, determine the formula for the volume of the tetrahedron bounded by the tangent hyper-plane and the positive $x$, $y$, $z$ and $w$-axes. Now do the same for the hyperboloid $xyzwt = k$ where $k$ is a constant. You may want to write this equation as $x_0x_1x_2x_3x_4 = k$. Now do the same with six variables, seven variables and then in general $n$ variables. Is there anything interesting about the formulas you derived? If so, what?

**Hint:** You may want to write the equations of your hyper-planes as

\[ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} = 1 \]

and in general

\[ \frac{x_0}{a_0} + \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} = 1 \]

The answers to the above questions must be completely typed using mathematical symbols where appropriate. Word and WordPerfect both have equation editing capabilities, as does Maple. Word and WordPerfect both allow you to copy and paste Maple graphics, commands and output into the word processor. Your answers should completely explain what you are doing and why you are doing it. They should look like good textbook explanations of the procedures and methodologies. You must include all appropriate graphs and Maple calculations. The questions dealing with visualizing two, three and four dimensions should all be addressed and should be done so in paragraph form. The six questions dealing with the areas and volumes of triangles to hyper-tetrahedrons should look more like Calculus textbook examples.