1. (10 Points) Find $f'(x)$ of

$$f(x) = 4x^3 + 3x^2$$

Solution:

$$f'(x) = 12x^2 + 2x3x^2 \ln(3)$$

2. (10 Points) Find $f'(x)$ of

$$f(x) = \sqrt{\ln(x^2 + x + 1)}$$

Solution:

$$f'(x) = \frac{1}{2} \left( \ln(x^2 + x + 1) \right)^{-1/2} \frac{2x + 1}{x^2 + x + 1}$$

3. (10 Points) Find where the following function is increasing, decreasing, concave up, and concave down. Find the critical numbers, inflection points and all relative maximums and minimums.

$$f(x) = 3x^4 - 16x^3 + 3$$

Solution: $f'(x) = 12x^3 - 48x^2 = 12x^2(x - 4)$, so the critical values are $x = 0$ and $x = 4$. Plotting these on the number line and determining the sign of the derivative in each subinterval gives.

| | + + + + + + + + + + |
|------------------------|
| 0                      |
| + + + + + + + + + + + + |
| 4                      |

Hence the function is decreasing on the interval $(-\infty, 4)$ and increasing on the interval $(4, \infty)$. So there is a relative minimum at $x = 4$. $f''(x) = 36x^2 - 96x = 12x(3x - 8)$, which is zero at $x = 0$ and $x = \frac{8}{3}$. Plotting these on the number line and determining the sign of the second derivative in each subinterval gives.

<table>
<thead>
<tr>
<th>+ + + + + + + + + +</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>+ + + + + + + + + + + +</td>
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<tr>
<td>$\frac{8}{3}$</td>
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Hence the function is concave up on the intervals $(-\infty, 0) \cup (\frac{8}{3}, \infty)$ and concave down on the interval $(0, \frac{8}{3})$. So there are points of inflection at $x = 0$ and $x = \frac{8}{3}$.

4. (10 Points) Find the following limit

$$\lim_{x \to \infty} \frac{x^2 - x + 1}{x^3 + 4x^2 - 2x + 5}$$

Solution:

$$\lim_{x \to \infty} \frac{x^2 - x + 1}{x^3 + 4x^2 - 2x + 5} = \lim_{x \to \infty} \frac{x^2 - x + 1}{x^3 + 4x^2 - 2x + 5} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^3}}{\frac{1}{x^3} + \frac{1}{x^3} + \frac{1}{x^3}}$$

$$= 0$$
5. (10 Points) Find the absolute maximum and absolute minimum (if they exist) of \( f(x) = x^4 - 8x^2 + 3 \) on \([-3,3]\).

**Solution:** \( f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2) \), so the critical values are at \(-2, 0 \) and \(2\). Evaluating the function at each of these and the endpoints gives: \( f(-3) = 12, f(-2) = -13, f(0) = 3, f(2) = -13, f(3) = 12 \). So the absolute maximum is 12 which is at both \( x = -3 \) and \( x = 3 \) and the absolute minimum is \(-13\) which is at both \( x = -2 \) and \( x = 2 \).

6. (10 Points) Use the second derivative test to determine all of the relative maximums and minimums of \( f(x) = x^4 - 8x^2 + 3 \).

**Solution:** \( f''(x) = 12x^2 - 16 \) and so \( f''(-2) = 32, f''(0) = -16 \) and \( f''(2) = 32 \). This implies that there is a relative maximum at \( x = 0 \) and relative minimums at both \( x = -2 \) and \( x = 2 \).

7. (10 Points) A fence is to be built around a 200-square-foot rectangular field. Three sides are to be made of wood costing $10 per foot, while the other side is made of stone costing $30 per foot. Find the dimensions of the enclosure that minimizes total cost.

**Solution:** Let the sides of the rectangle be labeled \( x \) and \( y \) then the cost of the fence is \( C = 10x + 30y + 10y = 40x + 20y \). Since the area must enclose 200 square feet we also have that \( xy = 200 \). Solving the second equation for \( y \) gives \( y = \frac{200}{x} \) and hence our cost function is \( C(x) = 40x + \frac{4000}{x} \). The interval of concern for \( x \) in this problem is \((0,\infty)\). Since the function approaches infinity as \( x \) approaches either 0 or infinity we know that the minimum will occur at a critical point. \( C'(x) = 40 - \frac{4000}{x^2} \) which is defined everywhere on \((0,\infty)\). The derivative is zero when \( x = 10 \) and the first derivative test confirms that this is indeed a minimum. So the dimensions that minimize the cost are \( x = 10 \) and \( y = 20 \).

8. (10 Points) Find the following integral,

\[
\int \sqrt{x}(x-1) \, dx
\]

**Solution:**
\[
\int \sqrt{x}(x-1) \, dx = \int x^{3/2} - x^{1/2} \, dx
\]
\[
= \frac{2}{3}x^{5/2} - \frac{2}{3}x^{3/2} + C
\]

9. (10 Points) Find the following integral,

\[
\int \frac{e^{-x} + 1}{e^{-x}} \, dx
\]

**Solution:**
\[
\int \frac{e^{-x} + 1}{e^{-x}} \, dx = \int 1 + e^{x} \, dx
\]
\[
= x + e^{x} + C
\]

10. (10 Points) Find the following integral,

\[
\int \frac{x}{\sqrt{2x^2 + 5}} \, dx
\]

**Solution:** Let \( u = 2x^2 + 5 \), then \( \frac{du}{dx} = 4x \) and hence \( dx = \frac{du}{4x} \). So
\[
\int \frac{x}{\sqrt{2x^2 + 5}} \, dx = \int \frac{x}{\sqrt{u}} \, du
\]
\[
= \frac{1}{4} \int u^{-1/2} \, du
\]
\[
= \frac{1}{4} \cdot \frac{5}{4} u^{4/5} + C
\]
\[
= \frac{5}{16} (2x^2 + 5)^{4/5} + C
\]
11. (Extra Credit) Find the following integral,

\[ \int \frac{x^3 + x^2 + x + 1}{3x^4 + 4x^3 + 6x^2 + 12x + 17} \, dx \]

**Solution:** Let \( u = 3x^4 + 4x^3 + 6x^2 + 12x + 17 \), then \( \frac{du}{dx} = 12x^3 + 12x^2 + 12x + 12 \). So \( dx = \frac{du}{12(x^3 + x^2 + x + 1)} \) and hence

\[ \int \frac{x^3 + x^2 + x + 1}{3x^4 + 4x^3 + 6x^2 + 12x + 17} \, dx = \int \frac{x^3 + x^2 + x + 1}{u} \, \frac{du}{12(x^3 + x^2 + x + 1)} = \frac{1}{12} \int \frac{1}{u} \, du = \frac{1}{12} \ln |u| + C = \frac{1}{12} \ln |3x^4 + 4x^3 + 6x^2 + 12x + 17| + C \]