Exam 2 Key

1. (10 points) Find $\frac{dy}{dx}
\begin{align*}
y &= \frac{3x^2 - 2x + 7}{(3x^3 - 2x)(2x - 1)}
\end{align*}
Solution:
\begin{align*}
\frac{dy}{dx} &= \frac{(3x^3 - 2x)(2x - 1)(6x - 2) - (3x^2 - 2x + 7)[2(3x^3 - 2x) + (9x^2 - 2)(2x - 1)]}{((3x^3 - 2x)(2x - 1))^2}
\end{align*}

2. (10 points) Find $\frac{dy}{dx}$
\begin{align*}
y &= e^{(-5x^3 + \frac{3}{x})}
\end{align*}
Solution:
\begin{align*}
\frac{dy}{dx} &= e^{(-5x^3 + \frac{3}{x})} \left(-15x^2 - \frac{3}{x^2}\right)
\end{align*}

3. (10 points) Find $\frac{dy}{dx}$
\begin{align*}
y &= 2^{3x^2}
\end{align*}
Solution:
\begin{align*}
\frac{dy}{dx} &= 2^{3x^2} \ln(2)3x^2 \ln(3)2x = 2 \ln(2) \ln(3)x3x^22^{3x^2}
\end{align*}

4. (10 points) Find the slope of the tangent line to the curve $f(x) = x^3 - 3x^2 + x - 2$ at the point $(1, -3)$.
Solution:
\begin{align*}
f'(x) &= 3x^2 - 6x + 1
\end{align*}
so the slope of the tangent line is $f'(1) = -2$

5. (10 points) Find the equation of the tangent line to the curve $f(x) = \sin(x)\cos(x)$ at $x = \frac{\pi}{4}$.
Solution:
\begin{align*}
f'(x) &= \cos^2(x) - \sin^2(x)
\end{align*}
so the slope of the tangent line is $f'\left(\frac{\pi}{4}\right) = \frac{1}{2} - \frac{1}{2} = 0$. So the equation of the tangent line is
\begin{align*}
y - f\left(\frac{\pi}{4}\right) &= f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) \\
y - \frac{1}{2} &= 0\left(x - \frac{\pi}{4}\right) \\
y &= \frac{1}{2}
\end{align*}
6. (10 points) Find \( \frac{dy}{dx} \)

\[ y^4 - 3xy^3 + \sin(xy) = 4x + 2y^2 \]

Solution:

\[
\begin{align*}
4y^3y' - 3(3xy^2y' + y^3) + \cos(xy)(xy' + y) &= 4 + 4y' \\
4y^3y' - 9xy^2y' - 3y^3 + xy' \cos(xy) + y \cos(xy) &= 4 + 4y' \\
4y^3y' - 9xy^2y' + xy' \cos(xy) - 4yy' &= 4 + 3y^3 - y \cos(xy) \\
y'(4y^3 - 9xy^2 + x \cos(xy) - 4y) &= 4 + 3y^3 - y \cos(xy) \\
y' &= \frac{4 + 3y^3 - y \cos(xy)}{4y^3 - 9xy^2 + x \cos(xy) - 4y}
\end{align*}
\]

7. (10 points) Find the equation of the tangent line to the curve \( y^2 - x^3 + 4xy = y^3 \) at the point \( (2, 1) \)

Solution:

\[
\begin{align*}
y^2 - x^3 + 4xy &= y^3 \\
2yy' - 3x^2 + 4xy' + 4y &= 3y^2y' \\
2yy' + 4xy' - 3y^2y' &= 3x^2 - 4y \\
y'(2y + 4x - 3y^2) &= 3x^2 - 4y \\
y' &= \frac{3x^2 - 4y}{2y + 4x - 3y^2}
\end{align*}
\]

So the slope of the tangent line to the curve at \( (2, 1) \) is

\[ m = \frac{3 \cdot 2^2 - 4 \cdot 1}{2 \cdot 1 + 4 \cdot 2 - 3 \cdot 1^2} = \frac{8}{7} \]

Hence the equation of the tangent line is

\[
\begin{align*}
y - 1 &= \frac{8}{7}(x - 2) \\
y &= \frac{8}{7}x - \frac{16}{7} + 1 \\
y &= \frac{8}{7}x - \frac{9}{7}
\end{align*}
\]

8. (10 points) The position function of a particle is given by

\[ s(t) = t^3 - \frac{9}{2}t^2 - 7t \quad t \geq 0 \]

When does the particle reach a velocity of 5 m/s? When does the particle reach a maximum velocity and what is that velocity?

Solution: The velocity of the particle is \( v(t) = s'(t) = 3t^2 - 9t - 7. \) So

\[
\begin{align*}
3t^2 - 9t - 7 &= 5 \\
3t^2 - 9t - 12 &= 0 \\
3(t - 4)(t + 1) &= 0
\end{align*}
\]

So either \( t = 4 \) or \( t = -1. \) Since \( t = -1 \) clearly makes no sense, the particle reaches a velocity of 5 m/s in 4 seconds. Since the velocity is given by a quadratic function that opens upward there will be no maximum velocity. There is, however, a minimum velocity and this will occur when \( t = \frac{3}{2} \) and that velocity is \( -\frac{55}{4} \) m/sec.
9. (10 points) If a ball is thrown vertically upward with a velocity of 80 ft/sec, then its height after \( t \) seconds is 
\[ s(t) = 80t - 16t^2. \]

(a) What is the maximum height reached by the ball?

**Solution:** The maximum height will be when the velocity is zero. Since \( v(t) = 80 - 32t \), the velocity will be zero when \( t = \frac{5}{2} \). At this moment the height of the ball will be \( s\left(\frac{5}{2}\right) = 100 \) feet.

(b) When is this maximum height reached?

**Solution:** As above, the maximum height will be when the velocity is zero. Since \( v(t) = 80 - 32t \), the velocity will be zero when \( t = \frac{5}{2} \).

(c) When does the ball hit the ground?

**Solution:** The ball will hit the ground when \( s(t) = 0 \). Solving \( 80t - 16t^2 = 0 \) gives \( t(80 - 16t) = 0 \) and hence \( t = 0 \) or \( t = 5 \). Clearly, \( t = 5 \) is the solution.

(d) What is the relationship between when the ball reaches its maximum height and when the ball hits the ground?

**Solution:** It takes twice as long for the ball to hit the ground as it does for the ball to reach its maximum height. In other words, the amount of time it takes for the ball to reach the top is the same as for the ball to fall back down.

10. (10 points) Do only one of the following exercises:

(a) (10 points) Using techniques similar to those used in class to show that 
\[ \frac{d}{dx} \left( \sin^{-1}(x) \right) = \frac{1}{\sqrt{1-x^2}}, \]
derive the formula for 
\[ \frac{d}{dx} \left( \cos^{-1}(x) \right). \]

**Solution:**

\[ y = \cos^{-1}(x) \]
\[
\cos(y) = x \\
-\sin(y)y' = 1
\]
\[
y' = -\frac{1}{\sin(y)}
\]
\[
y' = -\frac{1}{\sqrt{1-x^2}}
\]

(b) (10 points) Using the definition of the derivative show that 
\[ \frac{d}{dx} \left( \sin(x) \right) = \cos(x). \] You may use the fact that \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \) but all other calculations must be derived.

**Solution:**

\[
\frac{d}{dx} \left( \sin(x) \right) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}
\]
\[
= \lim_{h \to 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}
\]
\[
= \lim_{h \to 0} \frac{\sin(x) \cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x) \sin(h)}{h}
\]
\[
= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}
\]
\[
= \sin(x) \cdot 0 + \cos(x) \cdot 1
\]
\[
= \cos(x)
\]

Note:
\[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \lim_{h \to 0} \frac{\cos(h) - 1 \cos(h) + 1}{h \cos(h) + 1} \\
= \lim_{h \to 0} \frac{\cos^2(h) - 1}{h(\cos(h) + 1)} \\
= \lim_{h \to 0} \frac{-\sin^2(h)}{h(\cos(h) + 1)} \\
= -\lim_{h \to 0} \frac{\sin(h)}{\sin(h)} \frac{\sin(h)}{h \cos(h) + 1} \\
= -1 \cdot \frac{0}{2} \\
= 0
\]