1. (15 points) 

(a) Using \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) find the slope of the tangent line to \( f(x) = x^2 + 3x \) at the point \((-1, -2)\).

Solution:

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1} = \lim_{x \to -1} \frac{(x + 1)(x + 2)}{x + 1} = \lim_{x \to -1} x + 2 = 1
\]

(b) Using \( \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \) find the slope of the tangent line to \( f(x) = x^2 + 3x \) at the point \((-1, -2)\).

Solution:

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{(-1 + h)^2 + 3(-1 + h) + 2}{h} = \lim_{h \to 0} \frac{1 - 2h + h^2 - 3 + 3h + 2}{h} = \lim_{h \to 0} \frac{h^2 + h}{h} = \lim_{h \to 0} h + 1 = 1
\]

(c) Find the equation of the tangent line in the above exercises.

Solution:

\[
y + 2 = 1(x + 1)
y = x - 1
\]

2. (15 points) Using the definition of the derivative find an equation for the slope of the tangent line to \( f(x) = x^3 - 5x^2 + 1 \). Use the derivative to find the equation of the tangent line to the curve at \((1, -3)\).

Solution:

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^3 - 5(x + h)^2 + 1 - (x^3 - 5x^2 + 1)}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 5x^2 - 10xh + 5h^2 + 1 - x^3 + 5x^2 - 1}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 10xh + 5h^2}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh + h^2 - 10x + 5h}{h} = 3x^2 - 10x
\]

\[
y + 3 = -7(x - 1)
y = -7x + 4
\]
3. (10 points) Using the definition of the derivative find the derivative of \( f(x) = \frac{1}{\sqrt{x^2 + 2}} \).

Solution:

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{(x+h)^2 + 2}} - \frac{1}{\sqrt{x^2 + 2}}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{x^2 + 2} - \sqrt{(x+h)^2 + 2}}{h(\sqrt{(x+h)^2 + 2} + \sqrt{x^2 + 2})} \cdot \frac{\sqrt{x^2 + 2} + \sqrt{(x+h)^2 + 2}}{\sqrt{x^2 + 2} + \sqrt{(x+h)^2 + 2}}
\]

\[
= \lim_{h \to 0} \frac{x^2 + 2 - (x+h)^2}{h(\sqrt{(x+h)^2 + 2} + \sqrt{x^2 + 2})(\sqrt{x^2 + 2} + \sqrt{(x+h)^2 + 2})}
\]

\[
= \lim_{h \to 0} \frac{x^2 + 2 - x^2 - 2xh - h^2 - 2}{h(\sqrt{(x+h)^2 + 2} + \sqrt{x^2 + 2})(\sqrt{x^2 + 2} + \sqrt{(x+h)^2 + 2})}
\]

\[
= \lim_{h \to 0} \frac{-2xh - h^2}{h(\sqrt{(x+h)^2 + 2} + \sqrt{x^2 + 2})(\sqrt{x^2 + 2} + \sqrt{(x+h)^2 + 2})}
\]

\[
= \lim_{h \to 0} \frac{-2x - h}{\sqrt{(x+h)^2 + 2} + \sqrt{x^2 + 2}(\sqrt{x^2 + 2} + \sqrt{(x+h)^2 + 2})}
\]

\[
= \lim_{h \to 0} \frac{-2x}{\sqrt{(x+h)^2 + 2}(\sqrt{x^2 + 2} + \sqrt{(x+h)^2 + 2})}
\]

\[
= \lim_{h \to 0} \frac{-x}{(x^2 + 2)^{3/2}}
\]

4. (10 points) Using the definition of the derivative find the derivative of \( f(x) = \frac{x^2 - 1}{2 - x} \).

Solution:

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 1}{2-(x+h)} - \frac{x^2 - 1}{2-x}
\]

\[
= \lim_{h \to 0} \frac{(x+h)^2 - 1}{2-(x+h)} - \frac{x^2 - 1}{2-x}
\]

\[
= \lim_{h \to 0} \frac{(x+h)^2 - 1}{2-(x+h)} - \frac{x^2 - 1}{2-x}
\]

\[
= \lim_{h \to 0} \frac{(x+h)^2 - 1}{2-(x+h)} - \frac{x^2 - 1}{2-x}
\]

\[
= \lim_{h \to 0} \frac{2h^2 + 4xh + 2h^2 - 2 - x^3 - 2x^2h - xh^2 + x - 2x^2 + x^3 + x^2h + 2 - x - h}{h(2 - x)(2 - (x + h))}
\]

\[
= \lim_{h \to 0} \frac{4xh + 2h^2 - x^2h - xh^2 - h}{h(2 - x)(2 - (x + h))}
\]

\[
= \lim_{h \to 0} \frac{4x + 2h - x^2 - xh - 1}{(2 - x)(2 - (x + h))}
\]

\[
= \frac{4x - x^2 - 1}{(2 - x)(2 - x)}
\]

\[
= -\frac{x^2 + 4x - 1}{(2 - x)^2}
\]
5. (15 points) Using the differentiation rules find the derivatives of the following functions. Do not simplify your answer.

(a) \( f(y) = \left( \frac{1}{y^2} - \frac{3}{y^4} \right) (y + 5y^3) \)

Solution: 
\[
f'(y) = \left( \frac{1}{y^2} - \frac{3}{y^4} \right) (1 + 15y^2) + \left( -\frac{2}{y^3} + \frac{12}{y^5} \right) (y + 5y^3)
\]

(b) \( g(t) = \frac{t^2 + t + 1}{3t^2 - 2t + 1} \)

Solution: 
\[
g'(t) = \frac{(3t^2 - 2t + 1)(2t + 1) - (6t - 2)(t^2 + t + 1)}{(3t^2 - 2t + 1)^2}
\]

(c) \( h(t) = \frac{t e^t}{t^2 - e^t} \)

Solution: 
\[
h'(t) = \frac{\left( t^2 - e^t \right) (t e^t + e^t) - t e^t (2t - e^t)}{(t^2 - e^t)^2}
\]

6. (15 points) Find a cubic function
\[ y = ax^3 + bx^2 + cx + d \]
whose graph has horizontal tangents at \((-1, 4)\) and \((1, 7)\).

Solution: From the given information you get the following formulas
\[
3a - 2b + c = 0 \\
3a + 2b + c = 0 \\
-a + b - c + d = 4 \\
a + b + c + d = 7
\]

Manipulation of these equations gives \( a = -\frac{3}{4}, b = 0, c = \frac{9}{4} \) and \( d = \frac{11}{2} \). So the cubic is
\[ y = -\frac{3}{4} x^3 + \frac{9}{4} x + \frac{11}{2} \]

7. (10 points) Find the equations of the tangent lines to the curve
\[ y = \frac{x - 1}{x + 1} \]
that are parallel to the line \( x - 2y = 2 \).

Solution: The desired slope is \( \frac{1}{2} \) and
\[
f'(x) = \frac{2}{(x + 1)^2}
\]
Solving \( \frac{2}{(x + 1)^2} = \frac{1}{2} \) gives \( x = 1 \) or \( x = -3 \). So the tangent lines are \( y = \frac{1}{2} x - \frac{1}{2} \) and \( y = \frac{1}{2} x + \frac{7}{2} \).
8. (10 points) Newton’s Law of Gravitation says that the magnitude $F$ of the force exerted by a body of mass $m$ on a body of mass $M$ is

$$F = \frac{GmM}{r^2}$$

where $G$ is the gravitational constant and $r$ is the distance between the bodies. Find $\frac{dF}{dr}$ and explain its meaning. Suppose it is known that the Earth attracts an object with a force that decreases at a rate of 2 N/km when $r = 20,000$ km. How fast does the force change when $r = 10,000$ km?

**Solution:**

$$\frac{dF}{dr} = -\frac{2GmM}{r^3}$$

It represents the rate of change of the gravitational force between the two objects as the distance between the objects increases. If $F = 2$ N/km when $r = 20,000$ km, we obtain $GmM = 20000^3$ and hence

$$\left.\frac{dF}{dr}\right|_{r=10000} = -\frac{2 \cdot 20000^3}{10000^3} \text{ N/km} = -16 \text{ N/km}$$

9. (10 points extra credit) Using the product rule three times prove that if $f, g, h$ and $k$ are differentiable functions then $(fghk)' = f'ghk + fg'hk + fgh'k + fgkh'$. Use this formula to find

$$\frac{d}{dx} \left( e^x(x^5 - 4x^3 + x^2 - x - 1)(2x^3 + 7x^2 + 3x + 23) \left( \frac{x - 4}{x^2 - 2} - \frac{x^2 + 1}{x + 1} \right) \right)$$

**Solution:**

$$(fghk)' = (fg)h(k) + (fg)'h(k) + fgh(k)' + fghk'$$

With

- $f = e^x$, $f' = e^x$
- $g = x^5 - 4x^3 + x^2 - x - 1$, $g' = 5x^4 - 12x^2 + 2x - 1$
- $h = 2x^3 + 7x^2 + 3x + 23$, $h' = 6x^2 + 14x + 3$
- $k = \frac{x^4 - 4x^3 + x^2 - x + 1}{x^2 - 2}$, $k' = \frac{x^4 + 2x - 1}{(x^2 - 2)^2} - \frac{x^2 + 2x - 1}{(x + 1)^2}$

$$\frac{d}{dx} \left( e^x(x^5 - 4x^3 + x^2 - x - 1)(2x^3 + 7x^2 + 3x + 23) \left( \frac{x - 4}{x^2 - 2} - \frac{x^2 + 1}{x + 1} \right) \right)$$

$$= e^x(x^5 - 4x^3 + x^2 - x - 1)(2x^3 + 7x^2 + 3x + 23) \left( \frac{x - 4}{x^2 - 2} - \frac{x^2 + 1}{x + 1} \right)$$

$$+ e^x(5x^4 - 12x^2 + 2x - 1)(2x^3 + 7x^2 + 3x + 23) \left( \frac{x - 4}{x^2 - 2} - \frac{x^2 + 1}{x + 1} \right)$$

$$+ e^x(x^5 - 4x^3 + x^2 - x - 1)(6x^2 + 14x + 3) \left( \frac{x - 4}{x^2 - 2} - \frac{x^2 + 1}{x + 1} \right)$$

$$+ e^x(x^5 - 4x^3 + x^2 - x - 1)(2x^3 + 7x^2 + 3x + 23) \left( \frac{-x^2 + 8x - 2}{(x^2 - 2)^2} - \frac{x^2 + 2x - 1}{(x + 1)^2} \right)$$