Fall 2004

Exam 3 Key

1. Find \( \frac{d}{dx} (x^3) \).
   Solution: \( \frac{d}{dx} (x^3) = x^3 (\ln(x) + 1) \)

2. Find the derivative of \( f(x) = \sinh(x) \tanh(x) \).
   Solution: \( f'(x) = \cosh(x) \tanh(x) + \sinh(x) \sech^2(x) \)

3. Using differentials approximate the value of \( \sqrt[3]{28} \).
   Solution: \( \sqrt[3]{28} \approx \sqrt[3]{27} \cdot 1 + 1 \cdot \frac{1}{3} \cdot \sqrt[3]{27} = 3 + \frac{1}{27} = 3.037037037 \ldots \)

4. Find the absolute maximum and minimum of \( f(t) = t\sqrt{4-t^2} \) on the interval \([-1, 2]\).
   Solution: The absolute maximum is at \( t = \sqrt{2} \) and is \( f(\sqrt{2}) = 2 \). The absolute minimum is at \( t = -1 \) and is \( f(-1) = -\sqrt{3} \).

5. Show that the equation \( 1 + 2x + x^3 + 4x^5 = 0 \) has exactly one real solution.
   Solution: For notational purposes let \( f(x) = 1 + 2x + x^3 + 4x^5 \). First note that \( f(x) \) does have a real root, since \( f(0) = 1 \) and \( f(-1) = -6 \). If \( f(x) \) were to have more than one root then the function would need to change from increasing to decreasing at some point. Since \( f'(x) = 2 + 3x^2 + 20x^4 \), which is always positive, the function is always increasing. Hence there can be no more real roots.

6. Find \( \lim_{x \to \infty} \left( 1 + \frac{a}{x} \right)^{bx} \)
   Solution: \( \lim_{x \to \infty} \left( 1 + \frac{a}{x} \right)^{bx} = e^{ab} \)

7. For the function \( f(x) = \sqrt{x+4} \) find the intervals of increase or decrease, local maximums and minimum values, intervals of concavity and inflection points. Sketch a graph of the function.
   Solution: First note that \( f'(x) = \frac{4x+4}{3\sqrt{x^2}} \) and \( f''(x) = \frac{4x-8}{9\sqrt{x^2}} \). The function is increasing on the interval \((-1, \infty)\) and decreasing on the interval \((-\infty, -1)\). There is a local minimum at \( x = -1 \). The function is concave up on the intervals \((-\infty, 0) \cup (2, \infty)\) and concave down on the interval \((0, 2)\). There are points of inflection at \( x = 0 \) and \( x = 2 \). A sketch of the graph is below.
8. For the function

\[ f(x) = \frac{x^3 - 1}{x^3 + 1} \]

find the domain, intercepts, symmetries, asymptotes, intervals of increase or decrease, local maximums and minimum values, intervals of concavity and inflection points. Sketch a graph of the function.

**Solution:** First note that \( f'(x) = \frac{6x^2}{(x^3 + 1)^2} \) and \( f''(x) = \frac{-12x(2x^3 - 1)}{(x^3 + 1)^3} \). The domain of the function is all reals except for \( x = -1 \). The function has a y-intercept at \( y = -1 \) and an x-intercept at \( x = 1 \). The function is not symmetric with respect to the y-axis nor is it symmetric with respect to the origin. It has a vertical asymptote at \( x = -1 \) and a horizontal asymptote at \( y = 1 \). The function is increasing everywhere, more formally, on the intervals \((-\infty, -1) \cup (-1, \infty)\) and decreasing nowhere. There are no local minimums or maximums. The function is concave up on the intervals \((-\infty, -1) \cup (0, 2^{-1/3})\) and concave down on the intervals \((-1, 0) \cup (2^{-1/3}, \infty)\). There are points of inflection at \( x = 0 \) and \( x = 2^{-1/3} \). A sketch of the graph is below.