1. (5 Points Each) Find the following limits using algebraic methods.

(a) \( \lim_{x \to 2} \frac{x^4 - 16}{x - 2} \)

Solution:

\[
\lim_{x \to 2} \frac{x^4 - 16}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)(x^2 + 4)}{x - 2} = \lim_{x \to 2} (x + 2)(x^2 + 4) = 32
\]

(b) \( \lim_{x \to 1} \frac{\sqrt{x^2 + 8x} - 3}{x - 1} \)

Solution:

\[
\lim_{x \to 1} \frac{\sqrt{x^2 + 8x} - 3}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x^2 + 8x} - 3}{x - 1} \cdot \frac{\sqrt{x^2 + 8x} + 3}{\sqrt{x^2 + 8x} + 3} = \lim_{x \to 1} \frac{x^2 + 8x - 9}{(x - 1)(\sqrt{x^2 + 8x} + 3)} = \lim_{x \to 1} \frac{(x - 1)(x + 9)}{x + 9} = \lim_{x \to 1} \frac{x^2 + 8x + 3}{\sqrt{x^2 + 8x} + 3} = 5
\]

(c) \( \lim_{x \to \frac{\pi}{2}^+} \tan(x) \)

Solution: \( \lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty \)

(d) \( \lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 14}{-2x^3 - 7x + x^2 + 1} \)

Solution:

\[
\lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 14}{-2x^3 - 7x + x^2 + 1} = \lim_{x \to \infty} \frac{4x^3}{-2x^3} = \lim_{x \to \infty} \frac{4 - \frac{3}{x} + \frac{2}{x^2} - \frac{14}{x^3}}{-2 - \frac{7}{x^2} + \frac{1}{x} + \frac{1}{x^3}} = -2
\]
(e) \[ \lim_{x \to -\infty} \frac{x^2 - x + 1}{x - 5} \]

Solution:

\[
\lim_{x \to -\infty} \frac{x^2 - x + 1}{x - 5} = \lim_{x \to -\infty} \frac{\sqrt{x^2 - x + 1}}{x - 5} \cdot \frac{1}{\frac{1}{x}}
\]

\[
= \lim_{x \to -\infty} \frac{1}{x} \sqrt{\frac{1}{x^2}(x^2 - x + 1)}
\]

\[
= \lim_{x \to -\infty} \frac{1}{1 - \frac{5}{x}}
\]

\[
= \lim_{x \to -\infty} \frac{1 - \frac{1}{x} + \frac{1}{x}}{1 - \frac{5}{x}}
\]

\[
= -1
\]

2. (5 Points) Use the intermediate value theorem to prove that the function

\[ f(x) = x^4 - x^2 + x - 8 \]

has at least one real root.

**Solution:** Since \( f(x) \) is a continuous function and \( f(0) = -8 \) and \( f(2) = 6 \) the intermediate value theorem guarantees that there is a root to \( f(x) \) in the interval \((0, 2)\).

3. (5 Points) Find the horizontal and vertical asymptotes of the function

\[ f(x) = \frac{x^3 + 7x^2 - 3x + 6}{3x^4 - 6} \]

**Solution:** Since \( f(x) \) is a rational function, the vertical asymptotes are possibly at the values where the denominator is 0. Solving \( 3x^4 - 6 = 0 \) we get \( 3(x - \sqrt[4]{2})(x + \sqrt[4]{2})(x^2 + \sqrt{2}) = 0 \), and hence the real solutions are \( x = \sqrt[4]{2} \) and \( x = -\sqrt[4]{2} \). Since the numerator is not 0 at either of these two points we know that the one-sided limits as \( x \to \sqrt[4]{2} \) and \( x \to -\sqrt[4]{2} \) will be infinite, giving vertical asymptotes at \( x = \sqrt[4]{2} \) and \( x = -\sqrt[4]{2} \). For the horizontal asymptotes can be found by taking the limits as \( x \to \infty \) and \( x \to -\infty \),

\[
\lim_{x \to \infty} \frac{x^3 + 7x^2 - 3x + 6}{3x^4 - 6} = 0 \quad \lim_{x \to -\infty} \frac{x^3 + 7x^2 - 3x + 6}{3x^4 - 6} = 0
\]

So the horizontal asymptote is \( y = 0 \).
4. (10 Points) On the coordinate axes below draw the graph of a single function \( f(x) \) that has the following characteristics.

(a) \( \lim_{x \to -6^-} f(x) = -2 \)
(b) \( \lim_{x \to -6^+} f(x) = -1 \)
(c) \( f(-6) = 4 \)
(d) \( \lim_{x \to \infty} f(x) = 1 \)
(e) \( \lim_{x \to -\infty} f(x) = -3 \)
(f) \( \lim_{x \to 1^-} f(x) = \infty \)
(g) \( \lim_{x \to 1^+} f(x) = -\infty \)
(h) \( f'(8) = -1 \)
(i) \( f'(-3) = 1 \)

5. (5 Points) State the definition of the derivative of a function \( f(x) \).

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

if this limit exists.

6. (10 Points Each) Use the definition of the derivative to find the derivative of the following functions.

(a) \( f(x) = x^2 - x + 1 \)

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{(x + h)^2 - (x + h) + 1 - (x^2 - x + 1)}{h}
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h + 1 - x^2 + x - 1}{h}
= \lim_{h \to 0} \frac{2xh + h^2 - h}{h}
= \lim_{h \to 0} \frac{h(2x + h - 1)}{h}
= \lim_{h \to 0} 2x + h - 1
= 2x - 1
\]
(b) \( f(x) = \sqrt{3x^2 + x + 2} \)

Solution:

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 + (x+h) + 2} - \sqrt{3x^2 + x + 2}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{3(x+h)^2 + (x+h) + 2} - \sqrt{3x^2 + x + 2}}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 + (x+h) + 2 - (3x^2 + x + 2)}{h(\sqrt{3(x+h)^2 + (x+h) + 2} + \sqrt{3x^2 + x + 2})} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + x + h + 2 - 3x^2 - x - 2}{h(6x + 3h + 1)} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 + h}{h(6x + 3h + 1)} \\
&= \lim_{h \to 0} \frac{6x^2 + 6xh + 3h^2}{6x + 3h + 1} \\
&= \lim_{h \to 0} \frac{6x + 1}{2\sqrt{3x^2 + x + 2}} 
\end{align*}
\]

(c) \( f(x) = \frac{x}{x - 5} \)

Solution:

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{x + h}{x + h - 5} - \frac{x}{x - 5}}{h} \\
&= \lim_{h \to 0} \frac{h(\frac{(x+h)(x-5) - x(x+h-5)}{(x+h-5)(x-5)})}{h} \\
&= \lim_{h \to 0} \frac{x^2 + xh - 5x - 5h - x^2 - xh + 5x}{(x+h-5)(x-5)} \\
&= \lim_{h \to 0} \frac{-5h}{(x+h-5)(x-5)} \\
&= \lim_{h \to 0} \frac{-5}{(x+h-5)(x-5)} \\
&= \frac{-5}{(x-5)^2} 
\end{align*}
\]