1. (5 Points) State the definition of the derivative of a function $f(x)$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

if this limit exists.

2. (10 Points) Use the definition of the derivative to find the derivative of the following function.

$$f(x) = \sqrt{x^2 + 4x - 1}$$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^2 + 4(x+h) - 1} - \sqrt{x^2 + 4x - 1}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 4(x+h) - 1 - (x^2 + 4x - 1)}{h \left( \sqrt{(x+h)^2 + 4(x+h) - 1} + \sqrt{x^2 + 4x - 1} \right)}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 4x + 4h - 1 - x^2 - 4x + 1}{h \left( \sqrt{(x+h)^2 + 4(x+h) - 1} + \sqrt{x^2 + 4x - 1} \right)}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 4h}{h \left( \sqrt{(x+h)^2 + 4(x+h) - 1} + \sqrt{x^2 + 4x - 1} \right)}$$

$$= \lim_{h \to 0} \frac{2x + h + 4}{2 \sqrt{x^2 + 4x - 1} + \sqrt{x^2 + 4x - 1}}$$

$$= \lim_{h \to 0} \frac{2x + h + 4}{2x + 4}$$

$$= \frac{x + 2}{\sqrt{x^2 + 4x - 1}}$$

3. (5 Points Each) Find $f'(x)$ for each of the following functions. Do not simplify your answers.

(a) $f(x) = x^7 e^x$

Solution: $f'(x) = x^7 e^x + 7x^6 e^x$

(b) $f(x) = \sqrt{x^2 + 4x - 1}$

Solution: $f'(x) = \frac{1}{2} (x^2 + 4x - 1)^{-1/2} (2x + 4) = \frac{x + 2}{\sqrt{x^2 + 4x - 1}}$

(c) $f(x) = \frac{\cos(x)}{x^2 + x + 1}$

Solution: $f'(x) = \frac{-(x^2 + x + 1) \sin(x) - \cos(x)(2x + 1)}{(x^2 + x + 1)^2}$
(d) \( f(x) = \frac{\sqrt{x}}{\sin^2(x)} \)

Solution: 
\[
f'(x) = \frac{\sin^2(x) \frac{1}{2} x^{-4/5} - 2 \sin(x) \cos(x) \sqrt{x}}{\sin^4(x)}
\]

(e) \( f(x) = \frac{x^3 \sqrt{x^4 - 7x^2 + 1}}{(x + 1)^2 \sec(x)} \)

Solution:
\[
f(x) = \frac{(x + 1)^2 \sec(x)(x^3 \frac{1}{2}(x^4 - 7x^2 + 1)^{-1/2}(4x^3 - 14x) + 3x^2 \sqrt{x^4 - 7x^2 + 1} - (x + 1)^2 \sec(x) \tan(x) + 2(x + 1) \sec(x))(x^3 \sqrt{x^4 - 7x^2 + 1})}{((x + 1)^2 \sec(x))^2}
\]

(f) \( f(x) = \sqrt[3]{\frac{x \tan^{-1}(x)}{\ln(4x^2 + 3x)}} \)

Solution:
\[
f'(x) = \frac{1}{3} \left( \frac{x \tan^{-1}(x)}{\ln(4x^2 + 3x)} \right)^{-2/3} \left( \frac{\ln(4x^2 + 3x)}{\ln(4x^2 + 3x)} \right) - \frac{2x + 3x \ln(3)}{(\ln(4x^2 + 3x))^2} (x \tan^{-1}(x))
\]

4. (10 Points) Find all of the points where the following function has horizontal tangent lines.

\( f(x) = x^3 - 4x^2 + 3x - 1 \)

Solution: 
\( f'(x) = 3x^2 - 8x + 3 \), solving for \( x \) gives \( x = \frac{4 \pm \sqrt{7}}{3} \).

5. (10 Points) Find the equation of the tangent line to \( 2(x^2 + y^2)^2 = 25(x^2 - y^2) \) at the point (3, 1). A graph of the curve is below.

Solution: Differentiating implicitly gives \( 4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy') \). Substitute in the point (3, 1) to obtain \( 4(3^2 + 1^2)(2 \cdot 3 + 2 \cdot 1 \cdot y') = 25(2 \cdot 3 - 2 \cdot 1 \cdot y') \) which gives \( y' = -\frac{9}{13} \). So the tangent line equation is \( y = 1 - \frac{9}{13}(x - 3) = \frac{40}{13} - \frac{9}{13}x \).
6. (10 Points) Do only one of the following.

(a) Derive the formula for \( \frac{d}{dx} (\sin^{-1}(x)) \).

(b) Using the methods discussed in class prove \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \).

(c) Using \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \), prove that \( \lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0 \).

(d) Using \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \), \( \lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0 \) and the definition of the derivative prove that \( \frac{d}{dx} (\sin(x)) = \cos(x) \).

Solution: Proofs are in the text and/or were done in class.

7. (10 Points) Do only one of the following.

(a) State and prove the product rule.

(b) State and prove the quotient rule.

(c) State and prove the chain rule.

(d) Prove that \( \frac{d}{dx} (x^n) = nx^{n-1} \) for any real number \( n \).

Solution: Proofs are in the text and/or were done in class.