1. (10 points) Find the first and second derivatives of the following function

\[ f(x) = (1 - x^2)^{3/4} \]

Solution:

\[
\begin{align*}
f'(x) &= \frac{3}{4} (1 - x^2)^{-1/4} (-2x) \\
&= -\frac{3x}{2\sqrt{1 - x^2}}
\end{align*}
\]

\[
\begin{align*}
f''(x) &= \frac{3}{4} \left[ (1 - x^2)^{-1/4} (-2) - \frac{1}{4} (1 - x^2)^{-5/4} (-2x)(-2x) \right] \\
&= \frac{3x^2 - 6}{4 \sqrt{(1 - x^2)^5}}
\end{align*}
\]

2. (10 points) Find \( y'' \) of the following

\[ x^2 + xy + y^2 = 1 \]

Solution:

\[
\begin{align*}
2x + xy' + y + 2yy' &= 0 \\
y'(x + 2y) &= -y - 2x \\
y' &= \frac{-y + 2x}{x + 2y}
\end{align*}
\]

\[
\begin{align*}
y'' &= -\frac{(x + 2y) (y' + 2) - (y + 2x) (1 + 2y')}{(x + 2y)^2} \\
&= -\frac{(x + 2y) \left( \frac{-y + 2x}{x + 2y} + 2 \right) - (y + 2x) \left( 1 - \frac{2y + 2x}{x + 2y} \right)}{(x + 2y)^2} \\
&= -6 \left( \frac{x^2 + xy + y^2}{(x + 2y)^3} \right)
\end{align*}
\]

3. (10 points) Find \( f'(x) \) of the following

\[ f(x) = x^2 \ln (1 - x^2) \]

Solution:

\[
\begin{align*}
f'(x) &= x^2 \frac{-2x}{1 - x^2} + 2x \ln (1 - x^2) \\
&= \frac{-2x^3}{1 - x^2} + 2x \ln (1 - x^2)
\end{align*}
\]

4. (10 points) Find \( f'(x) \) of the following

\[ f(x) = x \sinh^{-1} \left( \frac{x}{3} \right) - \sqrt{9 + x^2} \]

1
\[
\begin{align*}
\text{Solution:} & \quad f'(x) = x \frac{1}{\sqrt{1 + \left(\frac{x}{3}\right)^2}} + \sinh^{-1}\left(\frac{x}{3}\right) - \left(\frac{1}{2} (9 + x^2)^{-1/2}(2x)\right) \\
& \quad = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9 + x^2}} - \frac{x}{\sqrt{9 + x^2}} \\
& \quad = \sinh^{-1}\left(\frac{x}{3}\right)
\end{align*}
\]

5. \textbf{(10 points)} Two cars start moving from the same point. One travels south at 60 mi/hr and the other travels west at 25 mi/hr. At what rate is the distance between the cars increasing two hours later?

\textbf{Solution:} Let \(x\) be the distance the car traveling west, \(y\) be the distance the first traveling south and \(z\) be the distance between the cars. By the Pythagorean theorem we have \(z^2 = x^2 + y^2\). Differentiating both sides with respect to time, \(t\).

\[
2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}
\]

So

\[
\frac{dz}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{z}
\]

In two hours, \(x = 50\), \(y = 120\) and hence \(z = \sqrt{50^2 + 120^2} = 130\). So

\[
\frac{dz}{dt} = \frac{50 \cdot 25 + 120 \cdot 60}{130} = 65
\]

6. \textbf{(10 points)} Find the absolute maximum and minimum of \(f(x) = x^2 + \frac{2}{x}\) on the interval \([\frac{1}{2}, 2]\).

\textbf{Solution:} First find the critical values.

\[
f'(x) = 2x - \frac{2}{x^2}
\]

So

\[
2x - \frac{2}{x^2} = 0
\]

\[
2x = \frac{2}{x^2}
\]

\[
x^3 = 1
\]

\[
x = 1
\]

Also, \(f'(x)\) does not exist when \(x = 0\), but this value is neither in the domain of \(f\) nor is it in the interval of interest. Check the values,

\[
f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \frac{2}{\frac{1}{2}} = 17
\]

\[
f(2) = 2^2 + \frac{2}{2} = 5
\]

\[
f(1) = 1^2 + \frac{2}{1} = 3
\]

So the minimum is 3 at \(x = 1\), and the maximum is 5 at \(x = 2\).
7. (10 points) Does there exist a function $f$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all $x$? If so, give an example, and if not, explain why.

Solution: No, since we wish $f'(x) \leq 2$ for all $x$, it is clear that the function must be continuous. In this case the mean value theorem applies, which says that there will be a point, $c$, in the interval $(0, 2)$ with

$$f'(c) = \frac{4 - (-1)}{2 - 0} = \frac{5}{2}$$

Since $\frac{5}{2}$ is greater than 2, it is impossible for $f'(x) \leq 2$ for all $x$.

8. (10 points) Find all the local maximums and minimums of the following function using both the first and second derivative tests.

$$f(x) = x + \sqrt{1 - x}$$

Solution:

$$f'(x) = 1 - \frac{1}{2}(1 - x)^{-1/2}$$

$$= 1 - \frac{1}{2\sqrt{1 - x}}$$

So $f'(x) = 0$ when

$$1 - \frac{1}{2\sqrt{1 - x}} = 0$$

$$\frac{1}{2\sqrt{1 - x}} = 1$$

$$2\sqrt{1 - x} = 1$$

$$\sqrt{1 - x} = \frac{1}{2}$$

$$1 - x = \frac{1}{4}$$

$$x = \frac{3}{4}$$

Also, $f'(x)$ does not exist when $x = 1$, but this is at the edge of the domain of $f$ and hence it is not a critical value. Using the first derivative test,

$$f'(0) = \frac{1}{2}, \quad f'(0.8) = 1 - \frac{1}{2\sqrt{1 - 0.8}} \approx -0.11803$$

Hence there is a local maximum at $x = \frac{3}{4}$. Using the second derivative test,

$$f''(x) = -\frac{1}{4\sqrt{(1 - x)^3}}$$

So

$$f''\left(\frac{3}{4}\right) = -\frac{1}{4\sqrt{(1 - \frac{3}{4})^3}} = -2$$

Again, this implies that there is a local maximum at $x = \frac{3}{4}$.

9. (10 points) Find the following limit

$$\lim_{x \to \infty} \left(x - \sqrt{x^2 - 1}\right)$$
10. (10 points) Extract all of the information you can from the following function. That is, find the domain, intercepts, symmetries, asymptotes, intervals of increase and decrease, concavity, local maximums and minimums, and points of inflection.

\[ f(x) = \frac{1}{x^3 - x} \]

**Solution:**

**Domain:** All real numbers such that \( x^3 - x \neq 0 \). That is, all real numbers except for \( x = 0, x = 1 \) and \( x = -1 \).

**Intercepts:** Since \( x = 0 \) is not in the domain of \( f \), there is no y-intercept. Also, since \( \frac{1}{x^3} = 0 \) gives \( 0 = 1 \), there are no x-intercepts either.

**Symmetry:** Since \( f(-x) = \frac{1}{(-x)^3 - (-x)} = -\frac{1}{x^3 - x} = -f(x) \) the function is odd and hence symmetric with respect to the origin.

**Asymptotes:** There are vertical asymptotes at \( x = 0, x = 1 \) and \( x = -1 \). Also, there is a horizontal asymptote at \( y = 0 \) since

\[ \lim_{x \to \infty} \frac{1}{x^3 - x} = 0 \]

**Intervals of Increase and Decrease:** First, find all of the critical values of the function,

\[ f'(x) = -\frac{3x^2 - 1}{(x^3 - x)^2} \]

So the critical values for \( f'(x) = 0 \), are where \( 3x^2 - 1 = 0 \), that is, \( x = \frac{1}{\sqrt{3}} \approx 0.57735 \) and \( x = -\frac{1}{\sqrt{3}} \approx -0.57735 \). The values where \( f'(x) \) does not exist are \( x = 0, x = 1 \) and \( x = -1 \). These are not critical values but they are places where the function can change direction. So the values of interest are \( x = -1, x = -\frac{1}{\sqrt{3}}, x = 0, x = \frac{1}{\sqrt{3}} \) and \( x = 1 \). Taking test values and evaluating, \( f'(-2) \approx -0.30556, f'(-0.75) \approx -6.3855, f'(-0.25) \approx 14.791, f'(0.25) \approx 14.791, f'(0.75) \approx -6.3855, \) and \( f'(2) \approx -0.30556 \). So the function is increasing on \((-\frac{1}{\sqrt{3}}, 0) \cup \left(0, \frac{1}{\sqrt{3}}\right)\) and decreasing on \((-\infty, -1) \cup (-1, -\frac{1}{\sqrt{3}}) \) and \(\left(-\frac{1}{\sqrt{3}}, 1\right) \cup (1, \infty)\).

**Local Maximums and Minimums:** From the analysis above, there is a local maximum at \( x = \frac{1}{\sqrt{3}} \) and a local minimum at \( x = -\frac{1}{\sqrt{3}} \).

**Concavity and Inflection Points:**

\[ f''(x) = \frac{12x^4 - 6x^2 + 2}{(x^3 - x)^3} \]

So \( f''(x) = 0 \) when \( 12x^4 - 6x^2 + 2 = 0 \), which has no real solutions since \( 36 - 4 \cdot 12 \cdot 2 = -60 \). So the only values of interest are \( x = 0, x = 1 \) and \( x = -1 \) (where the function does not exist). Taking test values and evaluating, \( f''(-2) = -0.78704, f''(-0.5) = 23.704, f''(0.5) = -23.704, f''(2) = 0.78704 \). Hence the intervals where the function is concave up are \((-1, 0) \) and \( (1, \infty) \) and the intervals where the function is concave down are \((-\infty, -1) \) and \((0, 1) \). There are no points of inflection since the function is undefined at \( x = 0, x = 1 \) and \( x = -1 \).