1. Evaluate the following limit

\[ \lim_{x \to 0} \left( \frac{1}{x} - \frac{3}{x^2 + 3x} \right) \]

\textbf{Answer:}

\[ \lim_{x \to 0} \left( \frac{1}{x} - \frac{3}{x^2 + 3x} \right) = \lim_{x \to 0} \left( \frac{x}{x^2 + 3x} \right) = \lim_{x \to 0} \left( \frac{1}{x + 3} \right) = \frac{1}{3} \]

2. In the theory of relativity, the mass of a particle with velocity \( v \) is

\[ m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \]

where \( m_0 \) is the mass of the object at rest and \( c \) is the speed of light. Find \( \lim_{v \to c^-} m \) and interpret the result. Why is a left-hand limit necessary?

\textbf{Answer:}

\[ \lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \infty \]

since as \( v \to c^- \), \( v^2 \to c^2 \) and hence \( 1 - \frac{v^2}{c^2} \to 0 \). So the numerator is a positive constant and the denominator is approaching 0, with all positive numbers, hence the quotient is getting big positively. An interpretation of this result is that to an outside observer the mass of an object will appear to grow without bound as the object’s speed increases to \( c \). The reason we must take a left-hand limit here is both mathematical and physical. In physics, the speed of an object with positive mass is limited to the speed of light and hence the velocity of the object will not exceed it, so \( c \) could not be approached from above. Mathematically, if \( v > c \) then \( 1 - \frac{v^2}{c^2} < 0 \) making the square root imaginary.

3. Find the constant \( c \) that makes \( g(x) \) continuous on \(( -\infty, \infty )\),

\[ g(x) = \begin{cases} \ x^2 - c^2 & \text{if} & x < 2 \\ \ cx + 5 & \text{if} & x \geq 2 \end{cases} \]

\textbf{Answer:} Clearly, each piece of this function is continuous for any constant \( c \). Hence we need only make sure that the two pieces are continuous at the point where the pieces switch, i.e. at \( x = 2 \). To ensure that the pieces match up at \( x = 2 \) all we must do is solve the equation \( 4 - c^2 = 2c + 5 \). This solution is \( c = -1 \).

4. Find the horizontal asymptote(s) to \( f(x) = \sqrt{4x^2 + x} - \sqrt{4x^2 - 7x} \).

\textbf{Answer:} To find the horizontal asymptote(s) we need to examine the two limits

\[ \lim_{x \to \infty} \sqrt{4x^2 + x} - \sqrt{4x^2 - 7x} \quad \text{and} \quad \lim_{x \to -\infty} \sqrt{4x^2 + x} - \sqrt{4x^2 - 7x} \]
\[
\lim_{x \to \infty} \sqrt{4x^2 + x} - \sqrt{4x^2 - 7x} = \lim_{x \to \infty} \sqrt{4x^2 + x - \sqrt{4x^2 - 7x} \cdot \frac{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}}{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}}} \\
= \lim_{x \to \infty} \frac{(4x^2 + x) - (4x^2 - 7x)}{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}} \\
= \lim_{x \to \infty} \frac{8x}{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}} \\
= \lim_{x \to \infty} \frac{8}{\sqrt{4 + \frac{1}{x}} + \sqrt{4 - \frac{7}{x}}} \\
= 2
\]

So we get the horizontal asymptote \( y = 2 \) from \( x \to \infty \).

\[
\lim_{x \to -\infty} \sqrt{4x^2 + x} - \sqrt{4x^2 - 7x} = \lim_{x \to -\infty} \sqrt{4x^2 + x - \sqrt{4x^2 - 7x} \cdot \frac{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}}{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}}} \\
= \lim_{x \to -\infty} \frac{(4x^2 + x) - (4x^2 - 7x)}{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}} \\
= \lim_{x \to -\infty} \frac{8x}{\sqrt{4x^2 + x} + \sqrt{4x^2 - 7x}} \\
= \lim_{x \to -\infty} \frac{8}{\sqrt{4 + \frac{1}{x}} + \sqrt{4 - \frac{7}{x}}} \\
= -2
\]

So we get the horizontal asymptote \( y = -2 \) from \( x \to -\infty \). Hence there are two horizontal asymptotes, one at \( y = -2 \) and one at \( y = 2 \).

5. Find the following limit

\[
\lim_{x \to \infty} \frac{2x^2 + x - 1}{-2x^3 + x^2 - 4x + 8}
\]

**Answer:**

\[
\lim_{x \to \infty} \frac{2x^2 + x - 1}{-2x^3 + x^2 - 4x + 8} = \lim_{x \to \infty} \frac{2x^2 + x - 1}{-2x^3 + x^2 - 4x + 8} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
= \lim_{x \to \infty} \frac{\frac{2}{x} + \frac{1}{x^2} - \frac{1}{x^3}}{-2 + \frac{1}{x} - \frac{4}{x^2} + \frac{8}{x^3}} \\
= 0
\]
6. Using the definition of the derivative, find \( f'(x) \) for \( f(x) = 2x^3 - 3 \).

**Answer:**

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{2(x + h)^3 - 3 - (2x^3 - 3)}{h}
= \lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 3 - 2x^3 + 3}{h}
= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h}
= \lim_{h \to 0} 6x^2 + 6xh + 2h^2
= 6x^2
\]

7. Using the definition of the derivative, find \( f'(x) \) for \( f(x) = \sqrt{x^2 - 1} \).

**Answer:**

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
= \lim_{h \to 0} \frac{\sqrt{(x + h)^2 - 1} - \sqrt{x^2 - 1}}{h}
= \lim_{h \to 0} \frac{(x + h)^2 - 1 - x^2 + 1}{h \left( (x + h)^2 - 1 + \sqrt{x^2 - 1} \right)}
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2 + 1}{h \left( (x + h)^2 - 1 + \sqrt{x^2 - 1} \right)}
= \lim_{h \to 0} \frac{2xh + h^2}{h \left( (x + h)^2 - 1 + \sqrt{x^2 - 1} \right)}
= \lim_{h \to 0} \frac{2x + h}{\sqrt{(x + h)^2 - 1 + \sqrt{x^2 - 1}}}
= \lim_{h \to 0} \frac{2x}{\sqrt{x^2 - 1}}
= \frac{2x}{\sqrt{x^2 - 1}}
\]

8. Prove that \( \lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0 \).

**Answer:** Recall that \(-1 \leq \sin(\pi/x) \leq 1\), and so \( e^{-1} \leq e^{\sin(\pi/x)} \leq e \), which gives us

\[
\frac{\sqrt{x}}{e} \leq \sqrt{x} e^{\sin(\pi/x)} \leq e\sqrt{x}
\]

and hence

\[
0 = \lim_{x \to 0^+} \frac{\sqrt{x}}{e} \leq \lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} \leq \lim_{x \to 0^+} e\sqrt{x} = 0
\]

so by the Squeeze Theorem we have

\[
\lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0
\]