1. (15 Points) Let the function $f(x)$ be defined by the following graph. Find the following limits and if a limit does not exist explain why.

![Graph of the function $f(x)$]

**Solution:**

(a) $\lim_{x \to -2^+} f(x) = 1$

(b) $\lim_{x \to -2^-} f(x) = \infty$, hence the limit does not exist since the function grows without bound to the left of $-2$.

(c) $\lim_{x \to 0^+} f(x) = 0$

(d) $\lim_{x \to 0^-} f(x) = 1$

(e) $\lim_{x \to 0} f(x)$ Does not exist since the one-sided limits are not equal to each other.

(f) $\lim_{x \to 2^+} f(x) = 3$

(g) $\lim_{x \to 2^-} f(x) = 1$

(h) $\lim_{x \to 5^+} f(x) \approx 1$

(i) $\lim_{x \to 5^-} f(x) \approx 1$

(j) $\lim_{x \to \infty} f(x)$ Does not exist since the function seems to oscillate between 1 and 3.

(k) $\lim_{x \to -\infty} f(x) = 0$

(l) List all points of discontinuity for $f(x)$. $-2, 0, 1,$ and $2$. 
2. (10 Points) On the coordinate axes below draw the graph of a function $f(x)$ that has the following characteristics.

(a) $\lim_{x \to -\infty} f(x) = \infty$ $\quad \lim_{x \to \infty} f(x) = 5$ $\quad \lim_{x \to -5^-} f(x) = \infty$ $\quad \lim_{x \to -5^+} f(x) = \infty$

(b) $\lim_{x \to -1} f(x) \text{ DNE} \quad \lim_{x \to -1^-} f(x) = 5$ $\quad \lim_{x \to -1^+} f(x) = 2$ $\quad \lim_{x \to 5} f(x) = 5$

(c) The function has vertical tangents at $x = -7$ and $x = 7$.

(d) The derivative does not exist at $x = 9$ but is continuous at $x = 9$.

Solution:

3. (5 Points) Use the intermediate value theorem to prove that the function $f(x) = x^3 - x^2 - 1$ has a root on the interval $(1, 2)$.

Solution: Since $f(1) = -1$ and $f(2) = 3$ the intermediate value theorem states that there exists a number $c \in (1, 2)$ such that $f(c) = 0$, hence $f(x)$ has a root on the interval $(1, 2)$.

4. (10 Points) Prove that

$$\lim_{x \to \infty} e^{-4x} \cos(x^2) = 0$$

Solution: Since $-1 \leq \cos(x^2) \leq 1$ we have $-e^{-4x} \leq e^{-4x} \cos(x^2) \leq e^{-4x}$. Since

$$\lim_{x \to \infty} -e^{-4x} = \lim_{x \to \infty} e^{-4x} = 0$$

the squeeze theorem implies that $\lim_{x \to \infty} e^{-4x} \cos(x^2) = 0$. 
5. (5 Points) Find the following limit.

\[
\lim_{x \to \pi \over 3} \sin(x/2) \tan(x)
\]

Solution:

\[
\lim_{x \to \pi \over 3} \sin(x/2) \tan(x) = \sin \left( \frac{\pi}{6} \right) \tan \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}
\]

6. (5 Points) Find the following limit.

\[
\lim_{x \to \infty} \frac{\sqrt{4x^4 - 3x^2 - x + 1}}{7x^2 - 2x + 8}
\]

Solution:

\[
\lim_{x \to \infty} \frac{\sqrt{4x^4 - 3x^2 - x + 1}}{7x^2 - 2x + 8} = \lim_{x \to \infty} \frac{\sqrt{4 - \frac{3}{x^2} - \frac{1}{x^3} + \frac{1}{x^4}}}{7 - \frac{2}{x} + \frac{8}{x^2}} = \frac{2}{7}
\]

7. (5 Points) State the definition of the derivative, as a function.

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

8. (10 Points Each) Using the definition of the derivative, as a function, find the derivative of the following functions.

   (a) \( f(x) = \frac{1}{x + 2} \)

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{1}{x + h + 2} - \frac{1}{x + 2}
\]

\[
= \lim_{h \to 0} \frac{x + 2 - (x + h + 2)}{h(x + h + 2)(x + 2)}
\]

\[
= \lim_{h \to 0} -\frac{h}{h(x + h + 2)(x + 2)}
\]

\[
= \lim_{h \to 0} \frac{-1}{(x + h + 2)(x + 2)}
\]

\[
= \frac{-1}{(x + 2)^2}
\]
(b) \( f(x) = \sqrt{x^2 - x} \)

Solution:

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
    &= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - (x+h)} - \sqrt{x^2 - x}}{h} \\
    &= \lim_{h \to 0} \frac{\sqrt{(x+h)^2 - (x+h)} - \sqrt{x^2 - x}}{h} \cdot \frac{\sqrt{(x+h)^2 - (x+h)} + \sqrt{x^2 - x}}{\sqrt{(x+h)^2 - (x+h)} + \sqrt{x^2 - x}} \\
    &= \lim_{h \to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h(\sqrt{(x+h)^2 - (x+h)} + \sqrt{x^2 - x})} \\
    &= \lim_{h \to 0} \frac{2x + 2xh + h^2 - x - h - x^2 + x}{h(\sqrt{(x+h)^2 - (x+h)} + \sqrt{x^2 - x})} \\
    &= \lim_{h \to 0} \frac{2x + h - 1}{\sqrt{(x+h)^2 - (x+h)} + \sqrt{x^2 - x}} \\
    &= \frac{2x - 1}{2\sqrt{x^2 - x}}
\end{align*}
\]

9. Extra Credit: (10 Points) Find the following limit using algebraic methods.

\[
\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}
\]

Solution:

\[
\begin{align*}
    \lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} &= \lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} \cdot \frac{\sqrt{6-x} + 2}{\sqrt{6-x} + 2} \\
    &= \lim_{x \to 2} \frac{6-x-4}{(\sqrt{3-x} - 1)(\sqrt{6-x} + 2)} \\
    &= \lim_{x \to 2} \frac{2-x}{(\sqrt{3-x} - 1)(\sqrt{6-x} + 2)} \\
    &= \lim_{x \to 2} \frac{2-x}{(\sqrt{3-x} - 1)(\sqrt{6-x} + 2)} \cdot \frac{\sqrt{3-x} + 1}{\sqrt{3-x} + 1} \\
    &= \lim_{x \to 2} \frac{(2-x)(\sqrt{3-x} + 1)}{(3-x-1)(\sqrt{6-x} + 2)} \\
    &= \lim_{x \to 2} \frac{(2-x)(\sqrt{3-x} + 1)}{(2-x)(\sqrt{6-x} + 2)} \\
    &= \frac{\sqrt{3-x} + 1}{\sqrt{6-x} + 2} \\
    &= \frac{1}{2}
\end{align*}
\]