1. (10 Points) Match the following differential equations with their direction fields.

(a) \( y' = \sin(y) + x^2 \)
(b) \( y' = e^{y-x} \)
(c) \( y' = x + y \)
(d) \( y' = y^2 \)

Solution:

2. (10 Points) The graph below is the direction field to the differential equation \( y' = (xy - 1)^2 - x^3 \). On the direction field, draw the graph of the solution to this differential equation with the initial condition \( y(0) = 1 \).

Solution:
3. (15 Points) Solve \( \frac{dy}{dt} = te^y \) under the initial condition \( y(1) = 0 \).

Solution:

\[
\frac{dy}{dt} = te^y \\
e^{-y} \frac{dy}{dt} = t \ dt \\
- e^{-y} = \frac{1}{2} t^2 + C \\
y = - \ln \left( C - \frac{1}{2} t^2 \right)
\]

Given that \( y(1) = 0 \), we have \( C = \frac{3}{2} \), and hence \( y = - \ln \left( \frac{3}{2} - \frac{1}{2} t^2 \right) \).

4. (15 Points) The half-life of cesium-137 is 30 years. Suppose we have a 50-mg sample of cesium-137. After how long will we have only 10 mg of cesium-137?

Solution: Using the half-life formula we have that the decay constant is \( k = -0.023104906 \), then solving the equation \( 10 = 50e^{-0.023104906t} \) gives \( t = 69.6578429 \) years.

5. (10 Points) What is the value of the continued fraction?

\[
1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}
\]

Solution: Considering the fractional portion of the expression we see that it is a solution to the equation \( x = \frac{1}{2+x} \), that is, \( 1 = 2x + x^2 \). The solutions to this equation are \( -1 \pm \sqrt{2} \). Adding back the 1 we see that the expression is either \( \sqrt{2} \) or \( -\sqrt{2} \). Since the expression is clearly positive it must be \( \sqrt{2} \).

6. (10 Points) What is the value of the following series?

\[
\sum_{n=1}^{\infty} \frac{(-4)^{n+2}}{7^n - 1}
\]

Solution:

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{(-4)^{n+2}}{7^n - 1} &= (-4)^3 + \frac{(-4)^4}{7} + \frac{(-4)^5}{7^2} + \cdots \\
&= (-4)^3 \left( 1 + \frac{4}{7} + \left( \frac{4}{7} \right)^2 + \cdots \right) \\
&= (-4)^3 \left( \frac{1}{1 - \frac{4}{7}} \right) \\
&= -\frac{448}{3}
\end{align*}
\]

7. (10 Points Each) Determine if the following series are absolutely convergent, conditionally convergent or divergent. Justify your conclusions and state the tests you employ.

(a) \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 25} \)

Solution: The series is conditionally convergent. If we let \( f(x) = \frac{x}{x^2 + 25} \) then \( f'(x) = \frac{25 - x^2}{(x^2 + 25)^2} \) which is negative for \( x > 5 \). Hence for \( n > 5 \) the sequence of terms is decreasing. Also, \( \lim_{n \to \infty} \frac{n}{n^2 + 25} = 0 \). So the series is convergent by the alternating series test. On the other hand, the series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 25} \) is divergent since by the limit comparison test with \( \sum_{n=1}^{\infty} \frac{1}{n} \) gives

\[
\lim_{n \to \infty} \frac{\frac{n}{n^2 + 25}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{n^2 + 25} = 1
\]
(b) \[ \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1} \]

**Solution:** The series is divergent. By the limit comparison test with \( \sum_{n=1}^{\infty} \frac{1}{n} \) gives

\[
\lim_{n \to \infty} \frac{n^2+1}{n^3+1} = \lim_{n \to \infty} \frac{n^3 + n}{n^3 + 1} = 1
\]

(c) \[ \sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}} \]

**Solution:** The series is convergent (and hence absolutely convergent). Using the root test we have

\[
\lim_{n \to \infty} \sqrt[n]{\frac{(2n)^n}{n^{2n}}} = \lim_{n \to \infty} \sqrt[n]{\left(\frac{2n}{n^2}\right)^n} = \lim_{n \to \infty} \frac{2n}{n^2} = 0
\]

(d) \[ \sum_{n=1}^{\infty} n^2 e^{-n^3} \]

**Solution:** The series is convergent (and hence absolutely convergent). Using the integral test we have

\[
\int x^2 e^{-x^3} \, dx = -\frac{1}{3} e^{-x^3}
\]

So

\[
\int_1^{\infty} x^2 e^{-x^3} \, dx = \lim_{t \to \infty} \int_1^t x^2 e^{-x^3} \, dx = -\frac{1}{3} \lim_{t \to \infty} e^{-t^3} - e^{-1} = \frac{1}{3e}
\]