1. \((15 \text{ Points})\) Determine if the following series converges or diverges. If it converges find its sum.

\[ \sum_{n=1}^{\infty} \left( \frac{1}{e^n} + \frac{1}{n(n+1)} \right) \]

**Solution:** The series converges and the sum is

\[ \sum_{n=1}^{\infty} \left( \frac{1}{e^n} + \frac{1}{n(n+1)} \right) = \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{e-1} + 1 = \frac{e}{e-1} \]

2. \((15 \text{ Points Each})\) Determine if the following series converge or diverge. Justify your answers.

(a) \(\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}\)

**Solution:** This series diverges by the \(n^{th}\) term test.

\[ \lim_{n \to \infty} \frac{2^n n!}{(n+2)!} = \lim_{n \to \infty} \frac{2^n}{(n+2)(n+1)} = \infty \]

(b) \(\sum_{n=2}^{\infty} \frac{1}{n \sqrt{\ln(n)}}\)

**Solution:** This series diverges by the integral test.

\[ \int_{2}^{\infty} \frac{1}{x \sqrt{\ln(x)}} \, dx = \lim_{t \to \infty} 2 \sqrt{\ln(x)} \bigg|_{2}^{t} = \infty \]

(c) \(\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}\)

**Solution:** This series converges by the ratio test.

\[ \lim_{n \to \infty} \frac{(n+1)^2 2^{n+1}}{n^2 2^n} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{(n+1)^2 2^{n+1}}{n^2 2^n} \frac{n!}{(n+1)(n+1)!} = \lim_{n \to \infty} \frac{2(n+1)}{n^2} = 0 \]

(d) \(\sum_{n=1}^{\infty} \frac{n + 2}{\sqrt{n^9 - n^4 + n - 1}}\)

**Solution:** This series converges by the limit comparison test with \(\sum_{n=1}^{\infty} \frac{1}{n^2}\).

\[ \lim_{n \to \infty} \frac{n+2}{\sqrt{n^9 - n^4 + n - 1}} = \lim_{n \to \infty} \frac{n^3 + 2n^2}{\sqrt{n^9 - n^4 + n - 1}} = \lim_{n \to \infty} \frac{n^3 + 2n^2}{\sqrt{n^9 - n^4 + n - 1}} \frac{1}{n^2} = \lim_{n \to \infty} \frac{1 + \frac{2}{n}}{1 - \frac{1}{n^2} + \frac{1}{n^5} - \frac{1}{n^7}} = 1 \]
3. \((15\text{ Points})\) Find the radius and interval of convergence of the following power series.

\[ \sum_{n=1}^{\infty} \frac{(3x - 2)^n}{5^n + 2} \]

\textbf{Solution:} Using the ratio test,

\[ \lim_{n \to \infty} \left| \frac{(3x - 2)^{n+1}}{5^{n+1} + 2} \right| = \frac{1}{5}|3x - 2| \]

so the power series will converge when \(-1 < \frac{1}{3}(3x - 2) < 1\) which gives \(-1 < x < \frac{7}{3}\). If we let \(x = -1\) the series becomes,

\[ \sum_{n=1}^{\infty} \frac{3(-1)^n}{5^n + 2} = \sum_{n=1}^{\infty} \frac{(-5)^n}{5^n + 2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{25} \]

which diverges by the \(n^\text{th}\) term test. If we let \(x = \frac{7}{3}\) the series becomes,

\[ \sum_{n=1}^{\infty} \frac{3\left(\frac{7}{3}\right)^n - 2}{5^n + 2} = \sum_{n=1}^{\infty} \frac{5^n}{5^n + 2} = \sum_{n=1}^{\infty} \frac{1}{25} \]

which also diverges by the \(n^\text{th}\) term test. So the interval of convergence is \((-1, \frac{7}{3})\) and the radius of convergence is \(R = \frac{5}{3}\).

4. \((15\text{ Points})\) Find the Taylor series expansion for \(f(x) = xe^x\) centered about \(a = 1\). Also find the radius and interval of convergence of the resulting series.

\textbf{Solution:} By taking several derivatives we find that \(f^{(n)}(x) = ne^x + xe^x\) so \(f^{(n)}(1) = (n + 1)e\) and hence the Taylor series expansion for \(f(x) = xe^x\) centered about \(a = 1\) is,

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n = \sum_{n=0}^{\infty} \frac{(n + 1)e}{n!} (x - 1)^n \]

To find the radius and interval of convergence we apply the ratio test

\[ \lim_{n \to \infty} \left| \frac{(n+2)e}{(n+1)!} (x - 1)^{n+1} \right| \leq \lim_{n \to \infty} \frac{(n + 2)}{(n + 1)^2} |x - 1| = 0 \]

so the radius of convergence is infinite and the interval of convergence is the entire real line.