1. This exercise will be dealing with the function \( f(x) = x^2(x + 1) \) on the interval \([0, 2]\).

(a) (10 Points) Using 4 rectangles find the right hand Riemann sum that approximates the area under the curve \( f(x) \) over the interval \([0, 2]\). Your answer must be correct to at least three decimal places.

Solution:
\[
\left( \frac{1}{2} \right)^2 \left( \frac{1}{2} + 1 \right) \cdot \frac{1}{2} + \left( 1 \right)^2 \left( 1 + 1 \right) \cdot \frac{1}{2} + \left( \frac{3}{2} \right)^2 \left( \frac{3}{2} + 1 \right) \cdot \frac{1}{2} + \left( 2 \right)^2 \left( 2 + 1 \right) \cdot \frac{1}{2} = 10
\]

(b) (15 Points) Using limit and summation notation, write an expression for the exact area under \( f(x) \) and over the interval \([0, 2]\). Then using sum and limit rules evaluate the limit.

Solution:
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\left( \frac{2i}{n} \right)^2}{\left( \frac{2i}{n} + 1 \right)} \cdot \frac{2}{n} = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left( \frac{8i^3}{n^3} + \frac{4i^2}{n^2} \right)
\]
\[
= \lim_{n \to \infty} \left( \frac{16}{n^2} \sum_{i=1}^{n} i^3 + \frac{8}{n^3} \sum_{i=1}^{n} i^2 \right)
\]
\[
= \lim_{n \to \infty} \left( \frac{16}{n^2} \left( \frac{n(n+1)}{2} \right)^2 + \frac{8}{n^3} \cdot n(n+1)(2n+1) \right)
\]
\[
= 4 + \frac{8}{3}
\]
\[
= \frac{20}{3}
\]

(c) (5 Points) Using the RiemannSum command in Maple, write the command that will find the right hand Riemann sum that approximates the area under the curve \( f(x) \) and above the interval \([0, 2]\).

Solution: \( \text{RiemannSum}(x^2*(x+1), x=0..2, \text{output}=\text{plot}, \text{method}=\text{right}, \text{partition}=4); \)

(d) (5 Points) Using the sum and limit commands in Maple, write the command that will find the exact area under the curve \( f(x) \) and above the interval \([0, 2]\).

Solution: \( \text{limit(\text{sum}((2*i/n)^2*(2*i/n+1)*2/n, i=1..n), n=infinity)}; \)

(e) (5 Points) Using the int command in Maple, write the command that will find the exact area under the curve \( f(x) \) and above the interval \([0, 2]\).

Solution: \( \text{int}(x^2*(x+1), x=0..2); \)

2. (7 Points Each) Using your integral rules and the Fundamental Theorem of Calculus evaluate the following.

(a) \( \int x^5 - \frac{3}{x} + \frac{7}{x^4} - \sqrt{x} + e^x \, dx \)

Solution:
\[
\int x^5 - \frac{3}{x} + \frac{7}{x^4} - \sqrt{x} + e^x \, dx = \frac{x^6}{6} - 3\ln|\, x| - \frac{7}{3x^3} - \frac{5}{6}x^{6/5} + e^x + C
\]
(b) \[ \int \frac{x}{\sqrt{2x^{2} - 5}} \, dx \]

**Solution:** Let \( u = 2x^{2} - 5, \)

\[ \int \frac{x}{\sqrt{2x^{2} - 5}} \, dx = \frac{1}{2} \sqrt{2x^{2} - 5} + C \]

(c) \[ \int x^{3} \sqrt{x^{3} - 1} \, dx \]

**Solution:** Let \( u = x^{3} - 1, \)

\[ \int x^{3} \sqrt{x^{3} - 1} \, dx = \frac{1}{3} \int u^{3/2} + u^{1/2} \, du = \frac{2}{15} (x^{3} - 1)^{5/2} + \frac{2}{9} (x^{3} - 1)^{3/2} + C \]

(d) \[ \int_{0}^{\pi/2} x^{2} \cos(x) \, dx \]

**Solution:**

\[ \int_{0}^{\pi/2} x^{2} \cos(x) \, dx = \left[ \frac{x^{3}}{3} - \sin(x) \right]_{0}^{\pi/2} = \frac{\pi^{3}}{24} - 1 \approx 0.291928196 \]

(e) \[ \frac{d}{dx} \int_{-7}^{\cos(x)} e^{-t^{2}} \sec \left( \sqrt{t} \right) \, dt \]

**Solution:** Let \( u = \cos(x), \) then \( du = -\sin(x) \) and

\[ \frac{d}{dx} \int_{-7}^{\cos(x)} e^{-t^{2}} \sec \left( \sqrt{t} \right) \, dt = \frac{d}{dx} \int_{-7}^{u} e^{-t^{2}} \sec \left( \sqrt{t} \right) \, dt = \frac{du}{dx} \cdot \frac{dy}{du} = -e^{-\cos^{2}(x)} \sec \left( \sqrt{\cos(x)} \right) \sin(x) \]

3. **(10 Points)** Find the area of the region bounded by the curves \( y = x^{3} - x \) and \( y = 5x \) to the right of the \( y \)-axis.

**Solution:** Setting the two functions equal to each other and solving we get \( 0, \sqrt{6}, -\sqrt{6}. \) Also to the right of the \( y \)-axis the function \( y = 5x \) is above \( y = x^{3} - x \) so the area of the region is

\[ \int_{0}^{\sqrt{6}} 5x \, dx = \left[ \frac{5x^{2}}{2} \right]_{0}^{\sqrt{6}} = 9 \]

4. **(5 Points)** Setup but do not evaluate the integral that will find the volume of the region bounded by \( y = \sqrt{25 - x^{2}}, y = 0, x = 2 \) and \( x = 4 \) rotated about the \( x \)-axis.

**Solution:**

\[ \int_{2}^{4} \pi \left( \sqrt{25 - x^{2}} \right)^{2} \, dx = \int_{2}^{4} \pi (25 - x^{2}) \, dx \]

5. **(5 Points)** Setup but do not evaluate the integral that will find the volume of the region bounded by \( y = \sqrt{25 - x^{2}}, y = 0, x = 2 \) and \( x = 4 \) rotated about the \( y \)-axis.

**Solution:**

\[ \int_{2}^{4} 2\pi x \sqrt{25 - x^{2}} \, dx \]

6. **(10 Points)** Find the volume of an object whose base is the triangle with vertices \((0, 0), (1, 0)\) and \((0, 1)\) and the cross sections perpendicular to the \( x \)-axis are semi-circles.

**Solution:** Taking a cross section at \( x \) forms a semi-circle with radius \( r = \frac{1-x}{2} \) and hence the area of a cross section is \( A(x) = \frac{1}{2} \pi r^{2} = \frac{1}{2} \pi \left( \frac{1-x}{2} \right)^{2} \). So the volume is

\[ V = \int_{0}^{1} \frac{1}{2} \pi \left( \frac{1-x}{2} \right)^{2} \, dx = \frac{\pi}{16} \]