1. Gabriel’s horn is constructed by taking the curve \( f(x) = \frac{1}{x} \) for \( 1 \leq x < \infty \) and rotating it about the \( x \)-axis. Find the volume and the surface area of the horn.

**Solution:** The volume is

\[
\int_1^\infty \pi \frac{1}{x^2} \, dx = \pi \lim_{t \to \infty} \frac{1}{t} \int_1^t \frac{1}{x^2} \, dx = -\pi \lim_{t \to \infty} \frac{1}{t} \left| \frac{1}{x} \right|_1^t = -\pi \lim_{t \to \infty} \left( \frac{1}{t} - 1 \right) = \pi
\]

The surface area is

\[
\int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx = 2\pi \lim_{t \to \infty} \frac{1}{t} \int_1^t \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx > 2\pi \lim_{t \to \infty} \frac{1}{t} \int_1^t \frac{1}{x} \, dx = 2\pi \lim_{t \to \infty} \ln(t) = \infty
\]

So the volume of Gabriel’s horn is finite but its surface area is infinite so you can fill it with paint but you can’t paint it.