1. (15 points) Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that first rotates points (counterclockwise) by $45^\circ$ and then reflects points through the horizontal axis. Find the standard matrix for $T$ and then use it to find $T\left(\begin{bmatrix} 2\sqrt{2} \\ -4\sqrt{2} \end{bmatrix}\right)$.

Solution: The standard matrix for $T$ is

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 2\sqrt{2} \\ -4\sqrt{2} \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} \\ -4\sqrt{2} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

2. (20 points) For the following matrices

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 0 \\ 5 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -1 & 8 \\ 0 & 2 & -3 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

find each of the following. If the calculation is not defined state why.

(a) $2A + 4B$

Solution:

$$2A + 4B = \begin{bmatrix} -4 & 2 \\ 28 & 26 \end{bmatrix}$$

(b) $E + 3D^T$

Solution:

$$E + 3D^T = \begin{bmatrix} 5 & 0 \\ -2 & 7 \\ 24 & -7 \end{bmatrix}$$

(c) $CD$

Solution: Does not exist. The number of columns of $C$ does not equal the number of rows of $D$.

(d) $CE$

Solution:

$$CE = \begin{bmatrix} 4 & 8 \\ 13 & 17 \\ 22 & 26 \end{bmatrix}$$

(e) $CD^T E^T$

Solution:

$$CD^T E^T = \begin{bmatrix} 46 & 18 & -10 \\ 94 & 39 & -16 \\ 142 & 60 & -22 \end{bmatrix}$$

(f) $A^2$

Solution:

$$A^2 = \begin{bmatrix} 8 & 9 \\ 36 & 53 \end{bmatrix}$$

(g) $D^2$

Solution: Does not exist since $D$ is not square.
3. (15 points) Use the determinant to decide which of the following matrices are invertible. If the matrix
is invertible, use the reduction technique to find its inverse.

\[ A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & -3 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \]

Solution:

\[ \det(A) = -1 \quad \det(B) = 0 \quad \det(C) = DNE \quad \det(D) = 6 \]

So A and D are the only ones that are invertible.

\[ A^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad D^{-1} = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ -1/6 & 1/3 & 1/6 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \]

4. (15 points) Use cofactor expansion to find the determinant of the following matrix. Once you are down
to 3 × 3 or smaller you may use the shortcut technique.

\[ \begin{bmatrix} 0 & 5 & -1 & 3 \\ 3 & -2 & 0 & 5 \\ 0 & 2 & 1 & 0 \\ 2 & -3 & 2 & 7 \end{bmatrix} \]

Solution:

\[ \begin{vmatrix} 0 & 5 & -1 & 3 \\ 3 & -2 & 0 & 5 \\ 0 & 2 & 1 & 0 \\ 2 & -3 & 2 & 7 \end{vmatrix} = -128 \]

5. (15 points) Use the reduction method to find the determinant of the following matrix. Once the matrix
is in upper-triangular form (but not before) you may finish the determinant.

\[ \begin{bmatrix} 0 & 5 & -1 & 3 \\ 3 & -2 & 0 & 5 \\ 0 & 2 & 1 & 0 \\ 2 & -3 & 2 & 7 \end{bmatrix} \]

Solution: Same as above.

6. (10 points) Use Cramer’s Rule to find the solutions to the following system of equations.

\[ 4x + 2y = 3 \]
\[ 7x - y = 5 \]

Solution: \( x = \frac{13}{18} \) and \( y = \frac{1}{18} \)

7. (10 points) Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices
at \((1,1,1), (-1,3,5)\) and \((2,-3,-1)\). Then find the volume of the parallelepiped after it has been sent
through the linear transformation \( T \) with standard matrix

\[ A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \]

Solution: The volume of the original parallelepiped is

\[ \left| \det \left( \begin{bmatrix} 1 & -1 & 2 \\ 1 & 3 & -3 \\ 1 & 5 & -1 \end{bmatrix} \right) \right| = |18| = 18 \]
The volume of the parallelepiped after it has been sent through the linear transformation $T$ is

$$\left| \det \left( \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \right) \right| \left| \det \left( \begin{bmatrix} 1 & -1 & 2 \\ 1 & 3 & -3 \\ 1 & 5 & -1 \end{bmatrix} \right) \right| = |6||18| = 108$$

8. (10 points) Mark each statement as either True or False.

(a) When two linear transformations are performed one after another, the combined effect may not always be a linear transformation. — **FALSE**

(b) If $A$ is a $3 \times 2$ matrix, then the transformation $x \mapsto Ax$ cannot be one-to-one. — **FALSE**

(c) If $A$ is a $3 \times 2$ matrix, then the transformation $x \mapsto Ax$ cannot be onto. — **TRUE**

(d) The product of invertible matrices is invertible. — **TRUE**

(e) If $A$, $B$ and $C$ are $n \times n$ matrices and if $ABC$ is invertible then $A$, $B$ and $C$ are all invertible. — **TRUE**

(f) If the linear transformation $x \mapsto Ax$ maps $\mathbb{R}^5$ onto $\mathbb{R}^5$ then $A$ has 5 pivot positions. — **TRUE**

(g) If the columns of an $n \times n$ matrix $A$ do not span $\mathbb{R}^n$ then the equation $Ax = b$ has an infinite number of solutions for each $b \in \mathbb{R}^n$. — **FALSE**

(h) $\det(AB) = \det(A) \det(B)$ — **TRUE**

(i) $\det(A + B) = \det(A) + \det(B)$ — **FALSE**

(j) $\det(A^T) = \det(A)$ — **TRUE**