1. **Definitions:** (5 Points Each) Give a definition for each of the following.

(a) A Linear Transformation — A Linear Transformation is a map \( T \) such that \( T(u + v) = T(u) + T(v) \) and \( T(cu) = cT(u) \).

(b) An Elementary Matrix — It is an \( n \times n \) matrix obtained from doing one elementary row operation on \( I_n \).

(c) A Subspace of \( \mathbb{R}^n \) — is any subset \( H \) of \( \mathbb{R}^n \) that has the properties
i. \( 0 \in H \),
ii. if \( u, v \in H \) then \( u + v \in H \),
iii. if \( u \in H \) then \( cu \in H \).

(d) The Null Space of a matrix — is the set of solutions to \( Ax = 0 \)

(e) A Basis to a subspace of \( \mathbb{R}^n \) — is a set of vectors that span the subspace and are linearly independent.

(f) The Rank of a matrix — is the number of basis vectors to the column space of the matrix. This is the same as the number of pivots in the matrix.

2. **True and False:** (2 Points Each) Mark each of the following as either true or false.

(a) Every matrix transformation is a linear transformation. — **True**

(b) A linear transformation preserves the operations of vector addition and scalar multiplication. — **True**

(c) A linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is completely determined by its effect on the columns of the \( n \times n \) identity matrix. — **True**

(d) The second row of \( AB \) is the second row of \( A \) multiplied on the right by the matrix \( B \). — **True**

(e) If \( A \) is an invertible \( n \times n \) matrix then the equation \( Ax = b \) has exactly one solution for each vector \( b \) in \( \mathbb{R}^n \). — **True**

3. **Theory:** (10 Points Each) Do each of the following.

(a) Show that if the columns of \( B \) are linearly dependent then so are the columns of \( AB \).

**Solution:** Let \( B = [b_1 \ b_2 \ \cdots \ b_n] \), then we have weights \( c_1, c_2, \ldots, c_n \), not all 0 with \( 0 = c_1b_1 + c_2b_2 + \cdots + c_nb_n \). Multiply both sides on the left by \( A \),

\[
0 = A0 = A(c_1b_1 + c_2b_2 + \cdots + c_nb_n)
\]

\[
= c_1Ab_1 + c_2Ab_2 + \cdots + c_nAb_n
\]

Since \( Ab_1, Ab_2, \ldots, Ab_n \) are the columns of \( AB \) this final equation shows that the columns of \( AB \) are dependent.
(b) Let $A$ be an $n \times n$ matrix whose columns are linearly independent. Show that the columns of $A^2$ span $\mathbb{R}^n$.

**Solution:** Since the columns of $A$ are linearly independent and $A$ is $n \times n$ we know that $A$ is invertible. Hence $A^2 = AA$ is also an invertible $n \times n$ matrix and therefore the columns of $A^2$ must span $\mathbb{R}^n$.

4. **Calculations:** (10 Points Each) Do each of the following.

(a) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation that sends $(3,0)$ to $(2,5)$ and $(1,2)$ to $(7,6)$. Find the matrix of $T$. Where does $T$ send the vector $(4,9)$? Is there a vector that gets sent to the vector $(4,9)$ by $T$? If so what is it?

**Solution:** Since $(2,5) = T(3,0) = 3T(1,0)$, so $T(1,0) = (2/3, 5/3)$. Also since $(7,6) = T(1,2) = T(1,0) + 2T(0,1)$, $2T(0,1) = (7,6) - (2/3, 5/3) = (19/3, 13/3)$ so $T(0,1) = (19/6, 13/6)$. Hence the matrix for $T$ is

$$A = \begin{bmatrix} 2/3 & 19/6 \\ 5/3 & 13/6 \end{bmatrix}$$

So $T(4,9)$ is

$$\begin{bmatrix} 2/3 & 19/6 \\ 5/3 & 13/6 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 187/6 \\ 197/6 \end{bmatrix}$$

To see if anything gets mapped to $(4,9)$ we simply solve,

$$\begin{bmatrix} 2/3 & 19/6 \\ 5/3 & 13/6 \end{bmatrix} \rightarrow \begin{bmatrix} 1/0 & 119/23 \\ 0/1 & 4/23 \end{bmatrix}$$

So the vector that gets sent to $(4,9)$ by $T$ is $(119/23, 4/23)$.

(b) Given

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 6 & -7 \\ 4 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -5 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Find each of the following products if they exist. If they do not exist state why.

i. $AB = \begin{bmatrix} 16 & 7 \\ -41 & 10 \\ 34 & 4 \end{bmatrix}$

ii. $BC = \begin{bmatrix} -33 & 3 & 15 \\ 9 & 6 & 0 \\ -15 & 0 & 6 \end{bmatrix}$

iii. $D^T D = \begin{bmatrix} 13 \end{bmatrix}$

iv. $DA$ — undefined since $D$ is $3 \times 1$ and $A$ is $3 \times 3$.

v. $CB = \begin{bmatrix} -29 & -5 \\ 14 & 8 \end{bmatrix}$

(c) Find $A^{-1}$ if it exists and if it does not explain why.

$$A = \begin{bmatrix} 7 & 12 & 4 \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -1 & 0 & 4 \\ 2 & -1 & -7 \\ -4 & 3 & 14 \end{bmatrix}$$
(d) Find bases for the column space and the null space of the following matrix.

\[ A = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 1 & 7 & 0 \\ 5 & 2 & 26 & -7 \end{bmatrix} \]

What is the rank of \( A \) and what is the nullity of \( A \)?

**Solution:**

\[ \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 1 & 7 & 0 \\ 5 & 2 & 26 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

So a basis for the column space is \{\((0,1,5),(1,1,2),(2,0,-7)\)\} and a basis for the null space is \{\((-4,-3,1,0)\)\}. Hence the rank is 3 and the nullity is 1.

(e) Let \( B = \{(-2,5,-1),(3,-1,1),(3,-2,1)\} \) and \( x = (1,2,3) \), find \([x]_B\).

**Solution:**

\[ \begin{bmatrix} -2 & 3 & 3 & 1 \\ 5 & -1 & -2 & 2 \\ -1 & 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -37 \end{bmatrix} \]

So, \([x]_B = (-8,32,-37)\).