1. Write the following equation in cylindrical coordinates and then write it in spherical coordinates.

\[ x^2 + y^2 - z^2 = 16 \]

**Solution:** Cylindrical

\[ r^2 - z^2 = 16 \]

**Spherical**

\[
\begin{align*}
\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta - \rho^2 \cos^2 \phi &= 16 \\
\rho^2 \sin^2 \phi - \rho^2 \cos^2 \phi &= 16 \\
\rho^2 (\sin^2 \phi - \cos^2 \phi) &= 16 \\
\rho^2 (1 - 2 \cos^2 \phi) &= 16
\end{align*}
\]

2. Find the volume of the solid bounded by the surface \( z = 6 - xy \) and the planes \( x = 2, x = -2, y = 0, y = 3 \) and \( z = 0 \).

**Solution:** The region we are integrating over is a rectangle with \( x \) ranging from -2 to 2 and \( y \) from 0 to 3. Hence the integral is

\[
\int_{-2}^{2} \int_{0}^{3} (6 - xy) \, dx \, dy = \int_{0}^{3} \left[ 6x - \frac{1}{2} x^2 y \right]_{-2}^{2} \, dy
\]

\[
= \int_{0}^{3} [12 - 2y - (-12 - 2y)] \, dy
\]

\[
= \int_{0}^{3} 24 \, dy
\]

\[
= 72
\]

3. Find the volume of the solid under the paraboloid \( z = 3x^2 + 2y^2 \) and above the region bounded by \( y = x^2 \) and \( x = y^2 \).

**Solution:** The region can be written as \( y \) ranging from \( x^2 \) to \( \sqrt{x} \) and \( x \) ranging from 0 to 1. Hence the integral is

\[
\int_{0}^{1} \int_{x^2}^{\sqrt{x}} 3x^2 + 2y^2 \, dy \, dx = \int_{0}^{1} \left[ 3x^2 y + \frac{2}{3} y^3 \right]_{x^2}^{\sqrt{x}} \, dx
\]

\[
= \int_{0}^{1} \left( 3x^2 \sqrt{x} + \frac{2}{3} (\sqrt{x})^3 \right) - \left( 3x^2 x^2 + \frac{2}{3} (x^2)^3 \right) \, dx
\]

\[
= \int_{0}^{1} 3x^{5/2} + \frac{2}{3} x^{3/2} - 3x^4 - \frac{2}{3} x^6 \, dx
\]

\[
= \left[ \frac{6}{7} x^{7/2} + \frac{4}{15} x^{5/2} - \frac{3}{5} x^5 - \frac{2}{21} x^7 \right]_{0}^{1}
\]

\[
= \frac{3}{7}
\]
4. Find the volume of the solid bounded by the paraboloid \( z = 10 - 3x^2 - 3y^2 \) and the plane \( z = 4 \).

**Solution:** This can be done with either a triple integral or a double integral, we will use a double integral. The region we are integrating over is a circle with radius \( \sqrt{2} \). This can be seen by solving for the intersection of the two surfaces.

\[
10 - 3x^2 - 3y^2 = 4 \\
3x^2 + 3y^2 = 6 \\
x^2 + y^2 = 2
\]

This region would be easier to handle in polar coordinates, so we will make that transition as well.

\[
\iint_R (10 - 3x^2 - 3y^2 - 4) \, dA = 3 \iint_R (2 - x^2 - y^2) \, dA \\
= 3 \int_0^{2\pi} \int_0^{\sqrt{2}} (2 - r^2) \, r \, dr \, d\theta \\
= 3 \int_0^{2\pi} \int_0^{\sqrt{2}} 2r - r^3 \, dr \, d\theta \\
= 3 \int_0^{2\pi} \left[ r^2 - \frac{1}{4} r^4 \right]_0^{\sqrt{2}} \, d\theta \\
= 3 \int_0^{2\pi} \, d\theta \\
= 6\pi
\]

5. Find

\[
\iiint_E z \, dV
\]

where \( E \) is bounded by the cylinder \( y^2 + z^2 = 9 \) and the planes \( x = 0, \, y = 3x \) and
Solution: There are a number of ways that this can be set up, here is one of them. We can look at the region as $x$ goes from 0 to $\frac{1}{3}y$, $z$ going from 0 to $\sqrt{\frac{9}{9} - y^2}$ and $y$ from 0 to 3. We could shift to polar coordinates after the first integral since the region is a quarter circle, but staying with rectangular coordinates is not too bad so we will stick with rectangular for the duration. Hence the integral is

\[
\int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \int_{0}^{\frac{1}{3}y} z \, dx \, dz \, dy = \int_{0}^{3} \int_{0}^{\sqrt{9-y^2}} \frac{1}{3} yz \, dz \, dy
\]

\[
= \frac{1}{3} \int_{0}^{3} y \left[ \frac{1}{2} z^2 \right]_0^{\sqrt{9-y^2}} \, dz \, dy
\]

\[
= \frac{1}{6} \int_{0}^{3} y (9 - y^2) \, dy
\]

\[
= \frac{1}{6} \int_{0}^{3} 9y - y^3 \, dy
\]

\[
= \frac{1}{6} \left[ \frac{9}{2} y^2 - \frac{1}{4} y^4 \right]_0^3
\]

\[
= \frac{27}{8}
\]

6. Evaluate

\[
\iiint_E x^2 + y^2 \, dV
\]

where $E$ is the hemispherical region that lies above the $xy$-plane and below the sphere $x^2 + y^2 + z^2 = 4$.

Solution: This is clearly a case for spherical coordinates since the region we are integrating over is a hemisphere. The hemisphere is of radius 2 and is the upper portion of the sphere, so the bounds will be $\rho$ from 0 to 2, $\theta$ from 0 to $2\pi$ and $\phi$ from
0 to $\frac{\pi}{2}$. Also note that the final integral requires a substitution for $\cos \phi$.

\[
\int_{E} x^2 + y^2 \, dV = \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \int_{0}^{2} \left( \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta \right) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]
\[
= \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \int_{0}^{2} \rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi
\]
\[
= \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \sin^3 \phi \left[ \frac{1}{5} \rho^5 \right]_0^2 \, d\theta \, d\phi
\]
\[
= \frac{32}{5} \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \sin^3 \phi \, d\theta \, d\phi
\]
\[
= \frac{64\pi}{5} \int_{0}^{\frac{\pi}{2}} \sin^3 \phi \, d\phi
\]
\[
= \frac{64\pi}{5} \int_{0}^{1} \left( 1 - \cos^2 \phi \right) \sin \phi \, d\phi
\]
\[
= \frac{64\pi}{5} \left[ u - \frac{1}{3}u^3 \right]_0^1
\]
\[
= 128\pi \frac{1}{15}
\]

7. Find

\[
\int_{R} x^2 - xy + y^2 \, dA
\]

where $R$ is the region bounded by the ellipse $x^2 - xy + y^2 = 2$. Use the transformation $x = \sqrt{2} \, u - \sqrt{2/3} \, v$, $y = \sqrt{2} \, u + \sqrt{2/3} \, v$ to do the integral.

**Solution:** Note that the function we are integrating is identical to the region we are integrating over, at least algebraically. Substituting the transformation into $x^2 - xy + y^2$ gives

\[
x^2 - xy + y^2 = \left( \sqrt{2}u - \sqrt{2/3}v \right)^2 - \left( \sqrt{2}u - \sqrt{2/3}v \right) \left( \sqrt{2}u + \sqrt{2/3}v \right) + \left( \sqrt{2}u + \sqrt{2/3}v \right)^2
\]
\[
= 2u^2 + 2v^2
\]

Hence the new region $S$ we are integrating over is $2u^2 + 2v^2 = 2$, in other words $u^2 + v^2 = 1$. Since this is a circle of radius one we will shift to polar coordinates after the first transformation is complete. To complete the first transformation we need the Jacobian for the transformation.

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}
\]
\[
= \begin{vmatrix}
\sqrt{2} & -\sqrt{2/3} \\
\sqrt{2} & \sqrt{2/3}
\end{vmatrix}
\]
\[
= \frac{4}{\sqrt{3}}
\]
Hence the integral is

\[
\iint_R x^2 - xy + y^2 \, dA = \frac{4}{\sqrt{3}} \iint_S 2u^2 + 2v^2 \, dA
\]

\[
= \frac{4}{\sqrt{3}} \int_0^{2\pi} \int_0^1 2r^2 \cdot r \, dr \, d\theta
\]

\[
= \frac{4}{\sqrt{3}} \int_0^{2\pi} \int_0^1 2r^3 \, dr \, d\theta
\]

\[
= \frac{4}{\sqrt{3}} \int_0^{2\pi} \left[ \frac{1}{2} r^4 \right]_0^1 \, d\theta
\]

\[
= \frac{2}{\sqrt{3}} \int_0^{2\pi} \, d\theta
\]

\[
= \frac{4\pi}{\sqrt{3}}
\]