Part 1:

1. Find the following limit if it exists, if it does not exist explain why.

\[\lim_{(x,y) \to (0,0)} \frac{xy^3}{x^2 + y^6}\]

**Solution:** Consider the path along the x-axis, that is, \(y = 0\). Along this path
\[
\frac{xy^3}{x^2 + y^6} = \frac{x0^3}{x^2 + 0^6} = 0
\]
Now consider the path along \(x = y^3\). Along this path
\[
\frac{xy^3}{x^2 + y^6} = \frac{y^3y^3}{(y^3)^2 + y^6} = \frac{y^6}{2y^6} = \frac{1}{2}
\]
Hence the limit does not exist.

2. Find the following limit if it exists, if it does not exist explain why.

\[\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}\]

**Solution:** First note that
\[-1 \leq \frac{x^2 - y^2}{x^2 + y^2} \leq 1\]
since
\[-(x^2 + y^2) = -x^2 - y^2 \leq -y^2 \leq x^2 - y^2 \leq x^2 + y^2\]
so
\[-xy \leq xy \frac{x^2 - y^2}{x^2 + y^2} \leq xy\]
and hence
\[0 = \lim_{(x,y) \to (0,0)} -xy \leq \lim_{(x,y) \to (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2} \leq \lim_{(x,y) \to (0,0)} xy = 0\]
Therefore,
\[\lim_{(x,y) \to (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0\]

Part 2:

1. Find all of the second partial derivatives of

\[f(x, y) = x^2 \sin(y) + \frac{x}{y^2}\]

**Solution:**
\[
fx(x, y) = 2x \sin(y) + \frac{1}{y^2}
\]
\[
f\phi(x, y) = x^2 \cos(y) - \frac{2x}{y^3}
\]
\[
f_{xx}(x, y) = 2 \sin(y)
\]
\[
f_{\phi\phi}(x, y) = -x^2 \sin(y) + \frac{6x}{y^4}
\]
\[
f_{x\phi}(x, y) = 2x \cos(y) - \frac{2}{y^3}
\]
2. Find \( \frac{\partial z}{\partial u} \) of the following.

\[ z = \cos(xy) \]

where

\[ x = v^2 - u^2 \quad y = v^2 + u^2 \]

Solution:

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\
= (-\sin(xy)y)(-2u) + (-\sin(xy)x)(2u) \\
= 2u\sin(xy)(y-x)
\]

Part 3:

1. Find the tangent plane to

\[ f(x,y) = x^2 \cos(y) + y^2 \cos(x) \quad \text{at} \quad \left( \frac{\pi}{4}, \frac{\pi}{3}, \pi^2 \left( \frac{1}{32} + \sqrt{2}/18 \right) \right) \]

Solution:

\[
f_x(x,y) = 2x \cos(y) - y^2 \sin(x) \\
f_y(x,y) = -x^2 \sin(y) + 2y \cos(x) \\
f_x \left( \frac{\pi}{4}, \frac{\pi}{3} \right) = 2 \frac{\pi}{4} \cos \left( \frac{\pi}{3} \right) - \left( \frac{\pi}{3} \right)^2 \sin \left( \frac{\pi}{4} \right) = \frac{\pi}{4} - \frac{\sqrt{2}\pi^2}{18} \\
f_y \left( \frac{\pi}{4}, \frac{\pi}{3} \right) = -\left( \frac{\pi}{4} \right)^2 \sin \left( \frac{\pi}{3} \right) + 2 \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}\pi}{3} - \frac{\sqrt{3}\pi^2}{32}
\]

so the tangent plane equation is

\[
z = \left( \frac{\pi}{4} - \frac{\sqrt{2}\pi^2}{18} \right) \left( x - \frac{\pi}{4} \right) + \left( \frac{\sqrt{2}\pi}{3} - \frac{\sqrt{3}\pi^2}{32} \right) \left( y - \frac{\pi}{3} \right) + \pi^2 \left( \frac{1}{32} + \sqrt{2}/18 \right)
\]

2. Find the tangent plane to

\[ f(x,y) = x^3y + xy^3 \quad \text{at} \quad (1, 2, 10) \]

Solution:

\[
f_x(x,y) = 3x^2y + y^3 \\
f_y(x,y) = x^3 + 3xy^2 \\
f_x(1,2) = 14 \\
f_y(1,2) = 13
\]

so the tangent plane equation is

\[
z = 14(x-1) + 13(y-2) + 10 = 14x + 13y - 30
\]

Part 4:

1. Find the directional derivative of

\[ f(x,y) = e^{-x}\sin(y) \]

at \((0, \frac{\pi}{3})\) in the direction of \((3, -2)\).
Solution: First, \( \nabla f = \langle -e^{-x}\sin(y), e^{-x}\cos(y) \rangle \), so \( \nabla f(0, \frac{\pi}{3}) = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle \). Also, the unit vector in the direction of \( \langle 3, -2 \rangle \) is \( \mathbf{u} = \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \). So

\[
D_u f = \nabla f \cdot \mathbf{u} = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \right\rangle = \frac{3\sqrt{3}}{2\sqrt{13}} - \frac{1}{\sqrt{13}} = \frac{3\sqrt{3} + 2}{2\sqrt{13}}
\]

2. Find \( \nabla f \) of

\( f(x, y) = y^2\tan^3(x) \)

at \( \left( \frac{\pi}{4}, -3 \right) \).

Solution: \( \nabla f = \langle 3y^2\tan^2(x)\sec^2(x), 2y\tan^3(x) \rangle \) so \( \nabla f \left( \frac{\pi}{4}, -3 \right) = (54, -6) \)

Part 5:

1. Locate all of the relative (local) maximums, minimums and saddle points of

\( z = 12xy - x^3 - y^3 \)

Several images of this surface are below.

![Surface Image 1](image1.png)  ![Surface Image 2](image2.png)

Solution:

\[
\begin{align*}
f_x(x, y) &= 12y - 3x^2 \\
f_y(x, y) &= 12x - 3y^2 \\
f_{xx}(x, y) &= -6x \\
f_{yy}(x, y) &= -6y \\
f_{xy}(x, y) &= 12
\end{align*}
\]

Setting \( f_x(x, y) = 0 \) gives \( y = \frac{1}{4}x^2 \). Setting \( f_y(x, y) = 0 \) and substituting \( \frac{1}{4}x^2 \) for \( y \) gives, \( x \left( 4 - \frac{x^2}{16} \right) = 0 \). Hence, either \( x = 0 \) or \( x = 4 \). So we have the critical points \((0, 0)\) and \((4, 4)\). Since \( D(x, y) = 36xy - 144, D(0, 0) = -144 \) and \( D(4, 4) = 432 \) with \( f_{xx}(4, 4) = -24 \). Thus we have a saddle point at \((0, 0)\) and a maximum at \((4, 4)\).

2. Locate the absolute maximums and minimums of

\( f(x, y) = e^{-(x^2+y^2+2x)} \)
over the region $D = \{(x, y) \mid x^2 + y^2 \leq 2\}$. The image below is the intersection of the cylinder $x^2 + y^2 = 2$ with the surface. The thick black line is the intersection of the two surfaces. Hence you are to find the absolute maximums and minimums of the surface on or inside this cylinder.

**Solution:** First we will find the critical points inside the domain.

$$f_x(x, y) = (-2x - 2)e^{-(x^2+y^2+2x)}$$
$$f_y(x, y) = -2ye^{-(x^2+y^2+2x)}$$

So the only critical point is $(-1, 0)$ which is in the domain. Now find the critical points on the boundary. The boundary is defined by

$$x = \sqrt{2}\cos(t)$$
$$y = \sqrt{2}\sin(t)$$
$$z = e^{-2-2\sqrt{2}\cos(t)}$$

for $0 \leq t \leq 2\pi$. So

$$\frac{dz}{dt} = 2\sqrt{2}\sin(t)e^{-2-2\sqrt{2}\cos(t)}$$

which gives two more critical points when $t = 0$ and $t = \pi$, these are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$. So

$$f(-1, 0) = e \approx 2.718281828$$
$$f(\sqrt{2}, 0) = e^{-(2+2\sqrt{2})} \approx 0.007999093$$
$$f(-\sqrt{2}, 0) = e^{-(2-2\sqrt{2})} \approx 2.289714471$$

Hence the maximum is at $(-1, 0)$ and the minimum is at $(\sqrt{2}, 0)$. 
