Exam #3 Key

1. **Short Answer: (4 Points Each)**: Answer all of the following.
   
   (a) Define a partial Turing computable function. — Page 131.
   
   (b) Define a Turing computable function. — Page 131.
   
   (c) Define a decidable language. — Page 132.
   
   (d) Define a Turing enumerable language. — Page 148.
   
   (e) State the Church-Turing thesis. — Page 122.

2. **True & False: (2 Points Each)** Mark each of the following as being either true or false.
   
   (a) A nondeterministic Turing machine is more powerful than a deterministic Turing machine. — False
   
   (b) A decidable language is Turing enumerable. — True.
   
   (c) A Turing machine with a random access read/write head is more powerful than a sequential head machine. — False.
   
   (d) The cardinality of the set of all Turing machines is uncountable. — False.
   
   (e) It is possible for a set to have the same cardinality as a proper subset of itself. — True.

3. **Turning Machines: (20 Points Each)**: Consider the following Turing machine.
   
   Consider the following Turing machine with starting state \( s \) and halting state \( h \).
   
   \[
   ((s, a), (s, \rightarrow)) \quad ((s, b), (q, \rightarrow)) \quad ((s, \sqcup), (t, \rightarrow)) \quad ((s, \triangleright), (s, \rightarrow))
   \]
   
   \[
   ((q, a), (s, \rightarrow)) \quad ((q, b), (h, b)) \quad ((q, \sqcup), (t, \rightarrow)) \quad ((q, \triangleright), (q, \rightarrow))
   \]
   
   \[
   ((t, a), (t, \rightarrow)) \quad ((t, b), (t, \rightarrow)) \quad ((t, \sqcup), (t, \rightarrow)) \quad ((t, \triangleright), (t, \rightarrow))
   \]

   Answer the following questions.
   
   i. What is the result with the initial tape of \( \triangleright aabaaabbabb \)?
   
      **Solution:** \( \triangleright aabaaabbabb \)
   
   ii. What is the result with the initial tape of \( \triangleright babb \)?
   
      **Solution:** \( \triangleright babb \)
   
   iii. What is the result with the initial tape of \( \triangleright aaba \)?
   
      **Solution:** Does not halt and does not alter the tape.
   
   iv. What is the result with the initial tape of \( \triangleright \sqcup \)?
   
      **Solution:** Does not halt and does not alter the tape.
   
   v. What does this Turing machine do?
   
      **Solution:** Does not alter the tape and halts on the second consecutive \( b \) if the string contains \( bb \). If the string does not contain \( bb \) then the machine does not halt. In other words, it semi-decides the language \( (a \cup b)^* bb (a \cup b)^* \).
(b) Using the primitives \( R, L, L_{\uparrow}, R_{\downarrow}, L_{\downarrow}, R_{\bar{0}}, L_{\bar{0}}, R_0, L_0, R_1, L_1, R_\bar{0}, L_\bar{0}, R_\bar{1}, L_\bar{1}, Shl, \) and \( Shr \) with the alphabet \( \{0,1,\uparrow,\downarrow\} \) construct a Turing machine (in diagram form) that takes an input number in binary form and subtracts one from it. Assume that the number is at the beginning of the tape and the read/write head is on the first space after the number and that the machine returns it to first space after the number before it halts. You may also assume that the number is not 0. For example, an input of \( \uparrow 101 \downarrow \) produces \( \uparrow 100 \downarrow \), an input of \( \uparrow 110 \downarrow \) produces \( \uparrow 101 \downarrow \), and an input of \( \uparrow 100 \downarrow \) produces \( \uparrow 11 \downarrow \).

Solution:

(c) Using the primitives \( R, L, L_{\uparrow}, L_{\downarrow}, R_{\downarrow}, L_{\downarrow}, R_\bar{0}, L_\bar{0}, R_0, L_0, R_1, L_1, R_\bar{0}, L_\bar{0}, R_\bar{1}, L_\bar{1}, Shl, Shr, A \) (add one), and \( S \) (subtract one) with the alphabet \( \{0,1,\uparrow,\downarrow\} \) construct a Turing machine (in diagram form) that takes two input numbers (in binary form) with a space between them and outputs their sum. Assume that the numbers are at the beginning of the tape and the read/write head is on the first character. You may also assume that the leftmost character of either number is 1 and hence neither number is 0. For example, an input of \( \uparrow 1011 \downarrow 111 \) produces \( \uparrow 10010 \downarrow \). The Turing machine \( S \) (subtract one) works like the subtraction machine in the previous exercise. The Turing machine \( A \) (add one) will add one to a number string given that the read/write head is on the space after the number and it returns the read/write head to the space after the number before it halts.

Solution:
4. **Infinity Proofs**: (10 Points) Do one and only one of the following.

(a) Prove that the rational numbers are countable. — Notes on Infinity pages 2–3.

(b) Prove that the real numbers are uncountable. — Notes on Infinity pages 4–5.

(c) Prove that the cardinality of the power set of a set $A$ is strictly greater than the cardinality of $A$. — Notes on Infinity page 5.

5. **Undecidability**: (10 Points) Do one and only one of the following.

(a) Prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ that is not partial Turing computable. — Pages 163–164.

(b) Let $H = \{ \langle M \rangle \langle w \rangle \mid M \text{ halts on } w \}$ and show that this language is not decidable. — Page 166.