Tangent Planes & Linear Approximations

Introduction and Goals:

The goals of this are two-fold. First we want you to become more familiar with the Maple commands concerning surfaces and planes. Second we want you to think about the transition from two-dimensions to three-dimensions with linear approximations. We will develop commands for finding the equations of tangent planes to surfaces defined as functions and surfaces that are defined implicitly. We will examine the concept of linear approximations in three-dimensions. Finally we will look at the TangentPlane command that is built into the VectorCalculus package.

Before You Start:

Make sure that you read and understand the mathematics from the corresponding sections in your textbook.

Textbook Correspondence:

Stewart 5th Edition: 15.4.

Maple Commands and Packages Used:

Packages: plots.
Commands: plot3d, implicitplot3d. limit, contourplot, spacecurve, seq, diff, D, solve, display.

History & Biographies:

Maple Commands:

Most of the commands in this lab are stored in the plots package; load it into your worksheet.

> with(plots):
Warning, the name changecoords has been redefined

As you from your reading the tangent plane equation for a surface that is defined as a function is
This first series of exercises will use Maple to find the tangent plane to the surface and then generalize this formula to create a command that will find the tangent plane to a surface at any point on the surface simply by giving it the function and the point of interest. In fact we will create two commands one of which will take the function and the point of tangency and return an expression for the tangent plane equation and another that will return the tangent plane equation as a function. Consider the function

\[ f(x, y) = x^2 - y^2 - \sin(xy) \]

Defining and graphing it gives us the following.

\[
> f:= (x, y) -> x^2 - y^2 - \sin(x*y);
\]

\[
> plot3d(f(x, y), x=-Pi..Pi, y=-Pi..Pi, axes=boxed);
\]

To find the tangent plane function we take the partial derivatives with respect to \( x \) and with respect to \( y \), evaluate them at the point of tangency, we will use \((1,2)\).

\[
> fx:= unapply(diff(f(x, y), x), x, y);
\]

\[
fx := (x, y) \rightarrow 2 x - \cos(y \cdot x) y
\]

\[
> fy:= unapply(diff(f(x, y), y), x, y);
\]

\[
fy := (x, y) \rightarrow -2 y - \cos(y \cdot x) x
\]

\[
> fx(1, 2);
\]

\[
2 - 2 \cos(2)
\]

\[
> fy(1, 2);
\]

\[-4 - \cos(2)\]
Then to construct the function we use the following command.

\[ f_{tp} := 2x(1, 2)(x-1) + 2y(1, 2)(y-2) + f(1, 2) \]

Plotting this plane with the original surface gives us the following image.

\[ \text{plot3d} \{ f(x, y), f_{tp} \}, x=-3..3, y=-3..3, \text{axes=boxed} \]
We can also create a command, TanPlaneFct that returns the tangent plane as a function simply by adding the unapply command to the plane definition, as below.

> TanPlaneFct := (f, xpos, ypos) -> unapply(unapply(diff(f(x, y), x), x, y)(xpos, ypos)*(x-xpos)+unapply(diff(f(x, y), y), x, y)(xpos, ypos)*(y-ypos)+f(xpos, ypos), x, y);

In this case we can set the tangent plane function to another variable name and evaluate the tangent plane at any point we wish. For example,

> tp := TanPlaneFct(f, 1, 1);

At this point would like to investigate how well the tangent plane approximates the surface close to the tangent point. Back in Calculus I you did the same thing with functions of a single variable by noticing how close the tangent line was to the curve, locally around a point of tangency. We are going to do exactly the same thing, except it will be in one more dimension. Also recall that in Calculus I, when we were dealing with functions of a single variable, we looked at the absolute value of the difference between the tangent line and the curve. We were interested in the interval about the point of
tangency that the absolute value of the difference was between 0 some tolerance. We will do the same thing here. We take the absolute value of the difference of the surface and the tangent plane and ask the question when is this absolute value less than a certain tolerance. With functions of a single variable, solving this equation usually gave us two points that bounded the region where the function and the tangent line were within the particular tolerance. When we’re dealing with surfaces and tangent planes we no longer get points, we get lines that bound regions in the plane. Getting exact solutions to these regions is far more difficult. We can however get an approximation to these regions using graphical techniques. For instance if we graph the absolute value of the difference between the function in the tangent plane along with the function \( z = 1 \) we get the following an image. By moving the surface around we can see what lies above what lies below the \( z = 1 \) surface giving us a very rough idea of the region where the surface is between 0 and 1. That is, the region where our original surface and tangent plane are within one unit of each other.

\[
> \text{plot3d}\{\text{abs}(f(x,y)-tp(x,y)),1\},x=-10..10,y=-10..10; \\
\]

For a better look at this region we can use the implicitplot function and graph the equation of the absolute value of the difference set equal to 1, our tolerance. This gives us the following image. Since our point of tangency was \((1,1)\) we can see that inside these curving lines is an approximation to the region where the tangent plane and the surface are within one unit of each other. As you can see from the image, representing these lines exactly using some type of parametric equations would be very difficult.

\[
> \text{implicitplot}(\text{abs}(f(x,y)-tp(x,y)))=1,x=-10..10,y=-10..10,grid=[100,100]); \\
\]
Zooming in on the same age will give us the following.

```maple
> implicitplot(abs(f(x,y)-tp(x,y))=1, x=-2..4, y=-2..4, grid=[100,100]);
```

Since coming up with the exact region is extremely difficult what we can do fairly easily is answer the question of how far away from the point of tangency can I get so that I will still be within one unit (our tolerance) of the surface? Even finding this radius exactly can be very difficult but we can do this graphically as well. Simply graph the same region along with a circle of radius $r$ where $r$ is predefined. For example, if we let $r = 1$ we get the following image. Notice that the circle does not lie entirely within the region and hence we cannot move one unit away from the point of tangency and still remain one unit from the surface.

```maple
> r:=1;

> display(implicitplot(abs(f(x,y)-tp(x,y))=1, x=-2..4, y=-2..4, grid=[100,100]), plot([r*cos(t)+1, r*sin(t)+1, t=0..2*Pi], color=black));
```
If we decrease the radius to $\frac{1}{2}$ we see that the circle is completely inside the region. Hence we can move $\frac{1}{2}$ unit from the point of tangency and remain within one unit of the surface but we can clearly go further.

\[
\begin{align*}
\texttt{\textgreater{} r := 0.5;} \\
\texttt{\textbackslash r := 0.5} \\
\texttt{\textgreater{} display(implicitplot(abs(f(x,y)-tp(x,y))=1,x=-2..4,y=-2..4,grid=[100,100]),plot([r*cos(t)+1,r*sin(t)+1,t=0..2*Pi],color=black));}
\end{align*}
\]

Trial and error show that at a radius of approximately 0.77 will give us the largest circle around the point of tangency that lies completely with inside the region. Hence if we are on the tangent plane at the tangent point we can move away from the point of tangency at most 0.77 units to remain within one unit of the surface.

\[
\begin{align*}
\texttt{\textgreater{} r := 0.77;} \\
\texttt{\textbackslash r := 0.77}
\end{align*}
\]
We can ask the same question with a different tolerance. How far can we get away from the point of tangency and still remain two units from the surface? Again trial and error gives us a radius of approximately 1.05.

> r := 1.05;

$r := 1.05$

How far can we get away from the point of tangency and remain $\frac{1}{2}$ unit away from the surface? Again trial and error gives us a radius of approximately 0.55.

> r := 0.55;

$r := 0.55$
One difficulty with the tangent plane equation generator commands that we created above is that they only work for surfaces that are defined as functions. If we have an implicitly defined surface we can no longer use these commands which makes sense since with implicitly defined services we use a different equation to create the tangent plane. Recall that this equation is given by

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

where the surface is represented by $F(x, y, z) = c$ with $c$ a constant. We will use the convention that the constant is on the right hand side of the defining equation. For example, say that we would like to find the tangent plane to the surface

$$x^2z + yz^2 + xy^2 = 2$$

at the point or points where $x = \frac{1}{2}$ and $y = 4$.

```plaintext
> g := x^2*z + y*z^2 + x^2*y = 2;
g := x^2*z + y*z^2 + x^2*y = 2

> implicitplot3d(g, x=-5..5, y=-5..5, z=-5..5, grid=[25,25,25], axes=boxed);
```
First we must find all of the values for $z$ that correspond to these values for $x$ and $y$. Using the solve command while substituting $x = \frac{1}{2}$ and $y = 4$ gives us two solutions for $z$. We will construct the tangent planes for both of these.

```maple
> solve(subs(x=1/2,y=4,g),z);
```

Next we take the partial derivatives of the left hand side of the implicitly defined surface with respect to $x$, $y$ and $z$. Note the use of Maple’s left hand side function lhs. It is imperative that all variables are on the left hand side of the surface definition and only constants are allowed to be on the right. Evaluating the partial derivatives at one of the points of tangency and plugging these into our plane equation gives us the following.

```maple
> gx:=unapply(diff(lhs(g),x),x,y,z);
  gx := (x, y, z) → 2 x z + 2 y x

> gy:=unapply(diff(lhs(g),y),x,y,z);
  gy := (x, y, z) → z^2 + x^2

> gz:=unapply(diff(lhs(g),z),x,y,z);
  gz := (x, y, z) → x^2 + 2 y z

> gtp1:=gx(1/2,4,-1/32+1/32*257^(1/2))*(x-1/2)+gy(1/2,4,-1/32+1/32*257^(1/2))*(y-4)+gz(1/2,4,-1/32+1/32*257^(1/2))*(z-(-1/32+1/32*257^(1/2)))=0;
```
Graphing this along with the original surface using the implicitplot3d command gives the following image.

Similarly for the other point of tangency we obtain.

> gtp2 := gx(1/2, 4, -1/32-1/32*257^(1/2)) * (x-1/2) + gy(1/2, 4, -1/32-1/32*257^(1/2)) * (y-4) + gz(1/2, 4, -1/32-1/32*257^(1/2)) * (z-(-1/32-1/32*257^(1/2))) = 0;

> implicitplot3d({g, gtp2}, x=-5..5, y=-5..5, z=-5..5, grid=[20, 20, 20], axes=boxed);
Graphing both together gives the following picture.

\[
\text{implicitplot3d(}\{g, gtp1, gtp2\}, x=-5..5, y=-5..5, z=-5..5, \text{grid=[20,20,20], axes=boxed});
\]

Now we would like to create a command that will give us the tangent plane equation given the surface and the \(x\), \(y\) and \(z\) coordinates of the point of tangency. Again we will make the stipulation that the surface is defined implicitly with only constants on the right hand side. We simply generalize the tangent plane equation to the following.

\[
\text{ImpTanPlane:=(g,xpos,ypos,zpos)->unapply(diff(lhs(g),x),x,y,z)(xpos,ypos,zpos)*(x-xpos)+unapply(diff(lhs(g),y),x,y,z)(xpos,ypos,zpos)*(y-ypos)+unapply(diff(lhs(g),z),x,y,z)(xpos,ypos,zpos)*(z-zpos)=0;}
\]
\[ \text{ImpTanPlane} := (g, \text{xpos}, \text{ypos}, \text{zpos}) \rightarrow \]
\[ \text{unapply} \left(\frac{d}{dx} \text{lhs}(g), x, y, z\right) \text{xpos, ypos, zpos} (x - \text{xpos}) + \text{unapply} \left(\frac{d}{dy} \text{lhs}(g), x, y, z\right) \text{xpos, ypos, zpos} (y - \text{ypos}) + \text{unapply} \left(\frac{d}{dz} \text{lhs}(g), x, y, z\right) \text{xpos, ypos, zpos} (z - \text{zpos}) = 0 \]

Using this command to find and graph the tangent plane to the surface

\[ x^2 z + yz^2 + x^2 y = 2 \]

at the point \((4,0,\frac{1}{8})\), we do the following sequence of commands.

\[ > \text{solve}(\text{subs}(x=4,y=0,g),z); \]
\[ \frac{1}{8} \]

\[ > \text{tp2} := \text{ImpTanPlane}(g,4,0,1/8); \]
\[ \text{tp2} := x - 6 + \frac{1025 y}{64} + 16 z = 0 \]

\[ > \text{implicitplot3d}([[g, \text{ImpTanPlane}(g,4,0,(1/8))], x=-5..5,y=-5..5,z=-5..5,grid=[20,20,20], axes=boxed]); \]

Similarly, we can find and graph the tangent planes we did originally by,

\[ > \text{solve}(\text{subs}(x=1/2,y=4,g),z); \]
\[ gtp1 := \text{ImpTanPlane}(g, 1/2, 4, -1/32 + 1/32 \times 257^{(1/2)}) \]

\[ gtp1 := -\frac{1}{32} + \frac{\sqrt{257}}{32}, -\frac{1}{32} - \frac{\sqrt{257}}{32} \]

\[ gtp1 : = \left( \frac{127}{32} + \frac{\sqrt{257}}{32} \right) \left( x - \frac{1}{2} \right) + \left( \left( -\frac{1}{32} + \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \right) (y - 4) + \frac{\sqrt{257}}{4} \left( z + \frac{1}{32} - \frac{\sqrt{257}}{32} \right) = 0 \]

\[ gtp1 := \text{ImpTanPlane}(g, 1/2, 4, -1/32 - 1/32 \times 257^{(1/2)}) \]

\[ gtp1 := \left( \frac{127}{32} - \frac{\sqrt{257}}{32} \right) \left( x - \frac{1}{2} \right) + \left( \left( -\frac{1}{32} - \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \right) (y - 4) - \frac{\sqrt{257}}{4} \left( z + \frac{1}{32} + \frac{\sqrt{257}}{32} \right) = 0 \]

\[ \text{implicitplot3d}({g, \text{ImpTanPlane}(g, 1/2, 4, -1/32 + 1/32 \times 257^{(1/2)})}), \text{x=-5..5, y=-5..5, z=-5..5, grid=[20,20,20], axes=boxed}); \]

\[ \text{implicitplot3d}({g, \text{ImpTanPlane}(g, 1/2, 4, -1/32 - 1/32 \times 257^{(1/2)})}), \text{x=-5..5, y=-5..5, z=-5..5, grid=[20,20,20], axes=boxed}); \]
Maple does have an internally defined function to create a tangent plane in the VectorCalculus package.

> with(VectorCalculus):
Warning, the assigned names <,> and <|> now have a global binding

Warning, these protected names have been redefined and unprotected: *, +, ., Vector, diff, int, limit, series
The difference between this function and the ones we created is that the surface needs be input in vector form, that is parametric form. If your surface is defined as a function then writing it in parametric form is very simple, simply let \( x = x \), \( y = y \) and \( z = f(x,y) \).

For example, to write the function \( f(x,y) = x^2 - y^2 - \sin(xy) \) parametrically we simply write, \( \langle x, y, x^2 - y^2 - \sin(xy) \rangle \). To define and graph this surface we execute the following commands.

\[
> \text{vf} := \langle x, y, x^2-y^2-\sin(x*y) \rangle;
\]
\[
> \text{plot3d(vf,x=-Pi..Pi,y=-Pi..Pi,axes=boxed);}
\]

The syntax of the TangentPlane command is simple, input the vector-valued function for the surface and the point of tangency given as equations, such as \( x = 1 \) and \( y = 1 \). In this example note that the tangent plane equation looks significantly different than the one that we derived above but it is the same plane only in vector form.

\[
> \text{TangentPlane(vf,x=1,y=1)};
\]
\[
> \text{plot3d({vf,TangentPlane(vf,x=1,y=1)},x=-Pi..Pi,y=-Pi..Pi,axes=boxed);} 
\]
If the surface is not given as a function but is written implicitly coming up with a general parameterization can be very difficult. We will see more examples of the parameterization of surfaces later in subsequent labs but here our surface is one that can be parameterized rather easily by noting that we can solve the equation for $y$.

\[
g := x^2 z + y z^2 + x^2 y = 2;  
\]

\[
solve(g, y);  
\]

\[
-\frac{x^2 z - 2}{z^2 + x^2}  
\]

\[
v := <x, \frac{-x^2 z - 2}{z^2 + x^2}, z>;  
\]

\[
vg := x e - \frac{x^2 z - 2}{z^2 + x^2} e + z e  
\]

Graphing this with the plot3d command and using the view option we get the following image. To see why we used the view option, remove it from the plot3d command and see the result.

\[
plot3d(vg, x=-5..5, z=-5..5, view=[-5..5,-5..5,-5..5], axes=box);  
\]
Using the TangentPlane command on the same points as we did before gives us the same tangent planes, except that they are represented in vector form.

\[ \text{TangentPlane}(vg, x=1/2, z=-1/32+1/32 \times 257^{1/2}) ; \]

\[
\begin{pmatrix}
\frac{1}{2} + x \\
\frac{1}{2} + x
\end{pmatrix} \mathbf{e}_x + \left( -\frac{257}{128} - \frac{\sqrt{257}}{128} \right) \left( \frac{1}{32} + \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \\
+ x \left( \frac{1}{32} + \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \left( -\frac{257}{128} + \frac{\sqrt{257}}{128} \right) \left( \frac{1}{32} + \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \\
+ \frac{1}{4} \left( -\frac{257}{128} + \frac{\sqrt{257}}{128} \right)^2 + \frac{1}{4} \left( \frac{1}{32} + \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \\
\left( -\frac{1}{32} + \frac{\sqrt{257}}{32} + z \right) \mathbf{e}_z
\]

\[ \text{TangentPlane}(vg, x=1/2, z=-1/32-1/32 \times 257^{1/2}) ; \]
\[
\begin{align*}
\left(\frac{1}{2} + x\right)e_x + & \left\{ -\frac{257}{128} - \frac{\sqrt{257}}{128} \right. \\
& \left. \frac{1}{32} - \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \\
+ x - & \frac{1}{32} - \frac{\sqrt{257}}{32} \\
& \left( \frac{1}{32} - \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \\
+ z - & \frac{1}{4} \left( \frac{1}{32} - \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \\
& \left( \frac{1}{32} - \frac{\sqrt{257}}{32} \right)^2 + \frac{1}{4} \\
\left( -\frac{1}{32} - \frac{\sqrt{257}}{32} + z \right)e_z
\end{align*}
\]

\begin{align*}
\text{plot3d} & \{vg, \text{TangentPlane}(vg, x=1/2, z=-1/32+1/32*257^{(1/2)})\} , x=-5..5, z=-5..5, \text{view}=[-5..5,-5..5,-5..5], \text{axes=box};
\end{align*}

\begin{align*}
\text{plot3d} & \{vg, \text{TangentPlane}(vg, x=1/2, z=-1/32-1/32*257^{(1/2)})\} , x=-5..5, z=-5..5, \text{view}=[-5..5,-5..5,-5..5], \text{axes=box};
\end{align*}
Exercises:

1. The following exercises concern the function $f(x, y) = \frac{x^2}{y^2 + 1}$.
   a. Graph this function on the region $[-5,5] \times [-5,5]$.
   b. Find the equation of the tangent plane to this surface at the point $(-2,3)$.
   c. Graph the surface and tangent plane to the surface at the point $(-2,3)$ on the same axes.
   d. Find the equation of the tangent plane to this surface at the point $(1,1)$. 
e. Graph the surface and tangent plane to the surface at the point \((1,1)\) on the same axes.
f. Create a plot of the curve that bounds the region in which the tangent plane is within a single unit of the surface.
g. What is the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the unit one region you found in the previous exercise?
h. Create a plot of the curve that bounds the region in which the tangent plane is within \(\frac{1}{2}\) unit of the surface.
i. What is the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the \(\frac{1}{2}\) unit region you found in the previous exercise?
j. Create a plot of the curve that bounds the region in which the tangent plane is within \(\frac{1}{4}\) unit of the surface.
k. What is the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the \(\frac{1}{4}\) unit region you found in the previous exercise?
l. Continue finding the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the \(\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}\) and \(\frac{1}{256}\) unit regions?
m. Graph the points you found in the previous exercises. Can you see a relationship between the scales and the radii? What does this imply about the surface as we decrease the tolerance?

2. The following exercises concern the function \(f(x, y) = \frac{\sin(e^x)}{\sin(e^y)} + 2\).

a. Graph this function on the region \([-5,5] \times [-5,5]\).
b. Find the equation of the tangent plane to this surface at the point \((-2,3)\).
c. Graph the surface and tangent plane to the surface at the point \((-2,3)\) on the same axes.
d. Find the equation of the tangent plane to this surface at the point \((1,1)\).

e. Graph the surface and tangent plane to the surface at the point \((1,1)\) on the same axes.
f. Create a plot of the curve that bounds the region in which the tangent plane is within \(1/10\) unit of the surface.
g. What is the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the \(1/10\) unit region you found in the previous exercise?
h. Create a plot of the curve that bounds the region in which the tangent plane is within \(1/50\) unit of the surface.
i. What is the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the \(1/50\) unit region you found in the previous exercise?
j. Create a plot of the curve that bounds the region in which the tangent plane is within \(1/100\) unit of the surface.
k. What is the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the \(1/100\) unit region you found in the previous exercise?
l. Continue finding the radius of the largest circle that can be centered at \((1,1)\) and is entirely contained within the \(1/200, 1/300, 1/400, 1/500, 1/700\) and \(1/1000\) unit regions?
m. Graph the points you found in the previous exercises. Can you see a relationship between the scales and the radii? What does this imply about the surface as we decrease the tolerance?

3. The following exercises concern the surface $|x|^3 = z^4 + xz^2 - yz + 1$.
   a. Graph this function on the region $[−5.5]×[−5.5]×[−5.5]$.
   b. Find all of the $z$ values that correspond to $x = 0$ and $y = 0$.
   c. Find the equations of the tangent planes to each of these points.
   d. Construct plots for each of the tangent planes and the original surface and then plot the original surface along with all of the tangent planes.

4. The total resistance $R$ of two resistors $R_1$ and $R_2$ connected in parallel is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.
   a. Graph this function on $[0,20]×[0,20]$.
   b. Say that we have two 8 ohm resistors connected in parallel. Find the linear approximation to the total resistance.
   c. Graph the region in which the linear approximation is within 1/100 of the actual combined resistance.
   d. If the two 8 ohm resistors fluctuate what type of fluctuation is best for the linear model. That is what is the relationship between the fluctuations of $R_1$ and $R_2$ that keep the approximated total resistance within 1/100 of the actual?
   e. If the two 8 ohm resistors fluctuate randomly what is the bound of their fluctuation that keeps the approximated total resistance within 1/100 of the actual?