Crash Course in Trigonometry

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Contents

Chapter 1

Trigonometric Functions

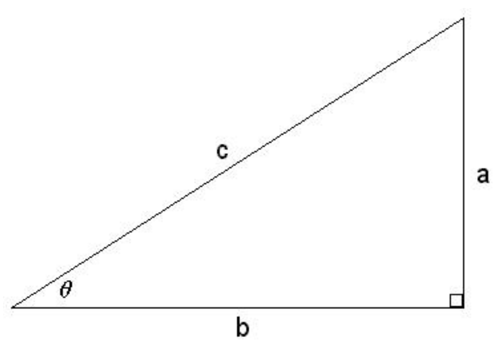
1.1 Introduction

Trigonometry is one of the oldest and most useful tools in mathematics. Trigonometry is used in surveying, forestry, mechanical engineering, modeling periodic phenomena, and the list goes on. We can look at trigonometry in three distinct but highly interconnected ways. We can consider trigonometry as a relation of the sides of a right triangle, called right triangle trigonometry. We can look at trigonometry as points on the unit circle, called unit circle trigonometry. The unit circle is a circle centered at the origin with radius one. Finally, we can look at trigonometry as functions on the Cartesian Coordinate System. All three of these formulations of trigonometry are actually equivalent. The right triangle formulation is probably the easiest and most intuitive of the three. Unit circle trigonometry is fairly easy to get the hang of if you understand the right triangle formulation and it is essential to the understanding of the function formulation. Thus, we will attack trigonometry in that order.

1.2 Right Triangle Trigonometry

1.2.1 Definitions

Throughout this discussion we will use the following right triangle as a notational reference.



We also represent the lengths of the sides of the triangle by their position relation to the angle θ. Specifically, the side labeled a is the side *opposite* the angle θ since it is on the other side of the triangle. For this reason it is sometimes labeled *opp*. The side labeled b is the side *adjacent* to the angle θ since it is on the same side of the triangle. For this reason it is sometimes labeled *adj*. The side labeled c is the *hypotenuse* and is sometimes labeled *hyp*.

There are six trigonometric functions that are defined as ratios of the lengths of this triangle. The sine of θ is denoted as $\sin(\theta)$ and it is the length of the opposite side a divided by the hypotenuse c, that is,

$$\sin(\theta) = \frac{opp}{hyp} = \frac{a}{c}$$

The cosine of θ is denoted as $\cos(\theta)$ and it is the length of the adjacent side b divided by the hypotenuse c, that is,

$$\cos(\theta) = \frac{adj}{hyp} = \frac{b}{c}$$

The remaining four trigonometric functions can be represented by the sine and the cosine as well as a, b and c. The tangent of θ is denoted as $\tan(\theta)$ and it is the length of the opposite side a divided by the length of the adjacent side b, that is,

$$\tan(\theta) = \frac{opp}{adj} = \frac{a}{b}$$

We can also represent it as the $\sin(\theta)$ divided by the $\cos(\theta)$. Note that with a little algebra we do obtain $\frac{a}{b}$ as well.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b}$$

The cotangent is the reciprocal of the tangent. So we can represent like that or by the cosine divided by the sine or by the adjacent side divided by the opposite side.

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{\frac{\sin(\theta)}{\cos(\theta)}} = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a}$$

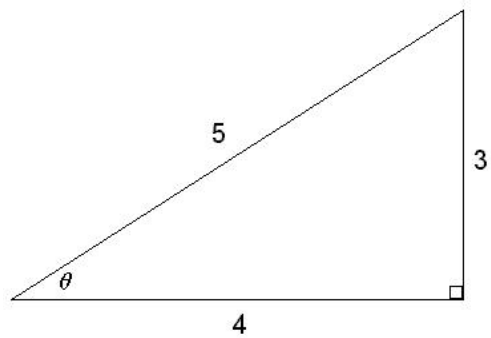
The secant of θ is the reciprocal of the cosine. So we can represent like that or by the hypotenuse divided by the adjacent side.

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\frac{b}{c}} = \frac{c}{b}$$

Finally, the cosecant of θ is the reciprocal of the sine. So we can represent like that or by the hypotenuse divided by the opposite side.

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{\frac{a}{c}} = \frac{c}{a}$$

Example 1.2.1 : Say we have the following triangle



A quick calculation using the Pythagorean theorem verifies that this is indeed a right triangle.

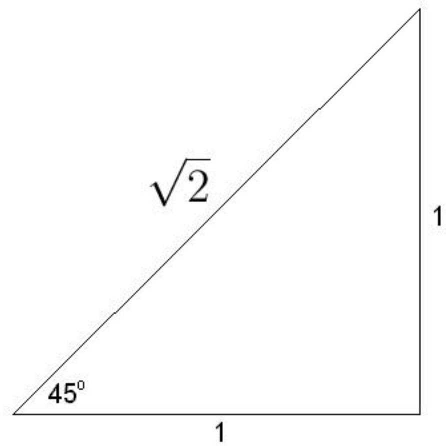
$$\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

The trigonometric functions on θ are as follows.

$$\begin{aligned}
\sin(\theta) &= \frac{a}{c} = \frac{3}{5} \\
\cos(\theta) &= \frac{b}{c} = \frac{4}{5} \\
\tan(\theta) &= \frac{a}{b} = \frac{3}{4} \\
\cot(\theta) &= \frac{b}{a} = \frac{4}{3} \\
\sec(\theta) &= \frac{c}{b} = \frac{5}{4} \\
\csc(\theta) &= \frac{c}{a} = \frac{5}{3}
\end{aligned}$$

1.2.2 Trigonometric Functions of Standard Angles

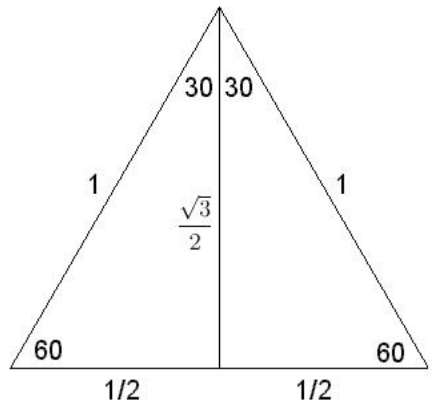
Most of the time when we want the sine or cosine of an angle we will get out our calculator and just type it in. Unless we have a TI-89 or 92, the calculator will give us an approximation to the value. In most cases this approximation is perfectly valid and in general finding the exact values of trigonometric functions can be rather difficult. There are some angles that, with a little work, we can find the exact values to the trigonometric functions. These will come in handy when we look at the other two formulations of trigonometry. The angles in question are 30°, 45° and 60°. We will call these the standard angles. First we will take care of the case of 45°.



Since the triangle is a 45-45-90 triangle we know that it is an isosceles triangle. Thus the legs a and b are both the same length and we will make them one. By the Pythagorean Theorem we now have that the hypotenuse has length $\sqrt{2}$. So the trigonometric functions of 45° are as follows,

$$
\begin{aligned}
\sin(45^\circ) &= \frac{a}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\cos(45^\circ) &= \frac{b}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\tan(45^\circ) &= \frac{a}{b} = \frac{1}{1} = 1 \\
\cot(45^\circ) &= \frac{b}{a} = \frac{1}{1} = 1 \\
\sec(45^\circ) &= \frac{c}{b} = \frac{\sqrt{2}}{1} = \sqrt{2} \\
\csc(45^\circ) &= \frac{c}{a} = \frac{\sqrt{2}}{1} = \sqrt{2}
\end{aligned}
$$

The calculations for the 30° and 60° angles are similar but we need a small geometric trick. Take the 30-60-90 triangle and sit it on end so that the 30° angle is pointing up. Now place a mirror reflection of the triangle right beside it. The two triangles together create a 60-60-60 triangle, that is, an equilateral triangle. So we know that all the sides of the big triangle are the same length and we can make them all one. Furthermore, where the two 30-60-90 triangles are connected is a perpendicular bisector, so we can divide the base into two segments of lengths $\frac{1}{2}$ and $\frac{1}{2}$. Again by the Pythagorean Theorem we have that the third side of the 30-60-90 triangle is $\frac{\sqrt{3}}{2}$. All in all we have the following diagram.



Now if we consider just one of the 30-60-90 triangles we have the following trigonometric functions for the 30° and 60° angles.

$$\begin{aligned}
\sin(30^\circ) &= \frac{a}{c} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \\
\cos(30^\circ) &= \frac{b}{c} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \\
\tan(30^\circ) &= \frac{a}{b} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\cot(30^\circ) &= \frac{b}{a} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \\
\sec(30^\circ) &= \frac{c}{b} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc(30^\circ) &= \frac{c}{a} = \frac{1}{\frac{1}{2}} = 2
\end{aligned}$$

and

$$\begin{aligned}
\sin(60^\circ) &= \frac{a}{c} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \\
\cos(60^\circ) &= \frac{b}{c} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \\
\tan(60^\circ) &= \frac{a}{b} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \\
\cot(60^\circ) &= \frac{b}{a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\sec(60^\circ) &= \frac{c}{b} = \frac{1}{\frac{1}{2}} = 2 \\
\csc(60^\circ) &= \frac{c}{a} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\end{aligned}$$

When we do the unit circle trigonometry we will expand upon this list. For the other angles that are not as nice as these we can simply use a calculator to find approximations.

1.2.3 Trigonometric Identities

Trigonometric identities are very useful whenever you are simplifying or solving trigonometric expressions. The nifty thing is that most of the identities come directly from the Pythagorean Theorem, and a little algebra.

First, $\sin^2(\theta) + \cos^2(\theta) = 1$ for any angle θ whatsoever. To see this just go back to the definitions of sine and cosine.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\
&= \frac{a^2}{c^2} + \frac{b^2}{c^2} \\
&= \frac{a^2 + b^2}{c^2} \\
&= \frac{c^2}{c^2} \\
&= 1
\end{aligned}$$

This also gives us

$$\sin^2(\theta) = 1 - \cos^2(\theta) \qquad \text{and} \qquad \cos^2(\theta) = 1 - \sin^2(\theta)$$

We can use these to get identities for the other four trigonometric functions as well. For example, take $\sin^2(\theta) = 1 - \cos^2(\theta)$ and divide both sides by $\cos^2(\theta)$

$$\begin{aligned}
\sin^2(\theta) &= 1 - \cos^2(\theta) \\
\frac{\sin^2(\theta)}{\cos^2(\theta)} &= \frac{1 - \cos^2(\theta)}{\cos^2(\theta)} \\
\tan^2(\theta) &= \frac{1}{\cos^2(\theta)} - \frac{\cos^2(\theta)}{\cos^2(\theta)} \\
\tan^2(\theta) &= \sec^2(\theta) - 1
\end{aligned}$$

Which gives

$$\sec^2(\theta) = \tan^2(\theta) + 1$$

Now take $\cos^2(\theta) = 1 - \sin^2(\theta)$ and divide both sides by $\sin^2(\theta)$.

$$\begin{aligned}
\cos^2(\theta) &= 1 - \sin^2(\theta) \\
\frac{\cos^2(\theta)}{\sin^2(\theta)} &= \frac{1 - \sin^2(\theta)}{\sin^2(\theta)} \\
\cot^2(\theta) &= \frac{1}{\sin^2(\theta)} - \frac{\sin^2(\theta)}{\sin^2(\theta)} \\
\cot^2(\theta) &= \csc^2(\theta) - 1
\end{aligned}$$

Which gives

$$\csc^2(\theta) = \cot^2(\theta) + 1$$

Summarizing,

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\sin^2(\theta) &= 1 - \cos^2(\theta) \\
\cos^2(\theta) &= 1 - \sin^2(\theta) \\
\sec^2(\theta) &= \tan^2(\theta) + 1 \\
\csc^2(\theta) &= \cot^2(\theta) + 1
\end{aligned}$$

1.3 Unit Circle Trigonometry

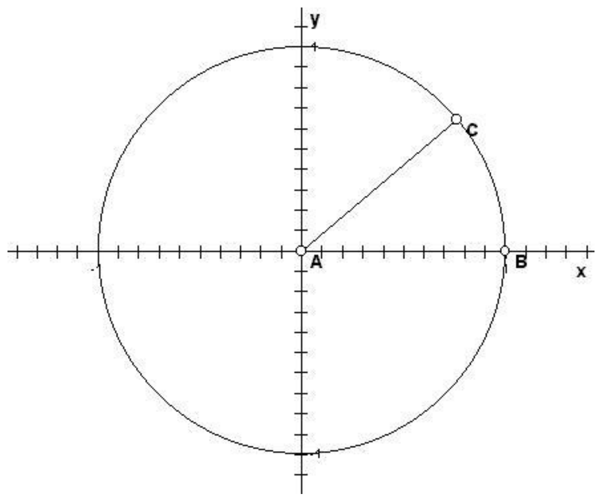
1.3.1 Setting up the Relationship

Let's move on to the unit circle formulation of trigonometry. When using the unit circle formulation (or looking at trigonometric functions on the Cartesian Coordinate System) we usually use radian measure rather than degree measurement. Radians are simply another unit for measuring the size of an angle. Just like feet and meters are two different ways to measure length. It turns out that there is an easy way to convert from degrees to radians and back. The formulas are,

$$x^\circ = x\left(\frac{\pi}{180}\right) \text{ radians} \qquad \text{and} \qquad x \text{ radians} = x\left(\frac{180}{\pi}\right) \text{ degrees}$$

We will see how these were derived shortly.

The radian measures the angle by using the length of the portion of the unit circle that is cut by the angle.



In other words, in radians, the measure of the angle $\angle CAB$ is the length of the arc BC. So if the length of BC was one then the angle $\angle CAB$ would be one radian. Since the circumference of a circle is $2\pi r$ and r in this case is 1, the length of the entire circle is 2π. So 2π is equivalent to 360°, which means

$$\begin{aligned} 2\pi \text{ radians} &= 360^\circ \\ 1 \text{ radian} &= \frac{360^\circ}{2\pi} \\ 1 \text{ radian} &= \frac{180^\circ}{\pi} \end{aligned}$$

and

$$\begin{aligned} 2\pi \text{ radians} &= 360^\circ \\ \frac{2\pi}{360} \text{ radians} &= 1^\circ \\ \frac{\pi}{180} \text{ radians} &= 1^\circ \end{aligned}$$

Now let's convert the standard angles to radians. Take 30°, this is $\frac{1}{12}$ of the entire circle. So in radians this would be $\frac{2\pi}{12} = \frac{\pi}{6}$. Now take 45°, this is $\frac{1}{8}$ of the entire circle. So in radians this would be $\frac{2\pi}{8} = \frac{\pi}{4}$. Finally, take $60°$, this is $\frac{1}{6}$ of the entire circle. So in radians this would be $\frac{2\pi}{6} = \frac{\pi}{3}$.

Degree	Radians
$30°$	$\frac{\pi}{6}$
$45°$	$\frac{\pi}{4}$
$60°$	$\frac{\pi}{3}$

We will now extend these angles around the circle.

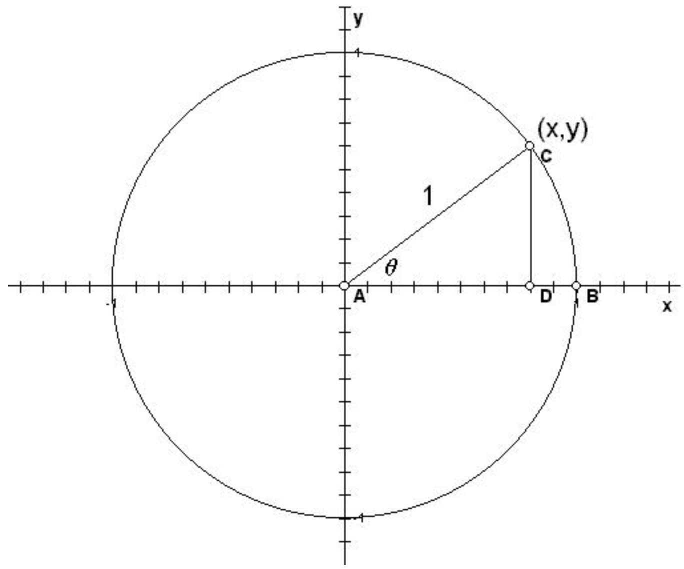
Degree	Radians
$0°$	0
$30°$	$\frac{\pi}{6}$
$45°$	$\frac{\pi}{4}$
$60°$	$\frac{\pi}{3}$
$90°$	$\frac{\pi}{2}$
$120°$	$\frac{2\pi}{3}$
$135°$	$\frac{3\pi}{4}$
$150°$	$\frac{5\pi}{6}$

Degree	Radians
$180°$	π
$210°$	$\frac{7\pi}{6}$
$225°$	$\frac{5\pi}{4}$
$240°$	$\frac{4\pi}{3}$
$270°$	$\frac{3\pi}{2}$
$300°$	$\frac{5\pi}{3}$
$315°$	$\frac{7\pi}{4}$
$330°$	$\frac{11\pi}{6}$

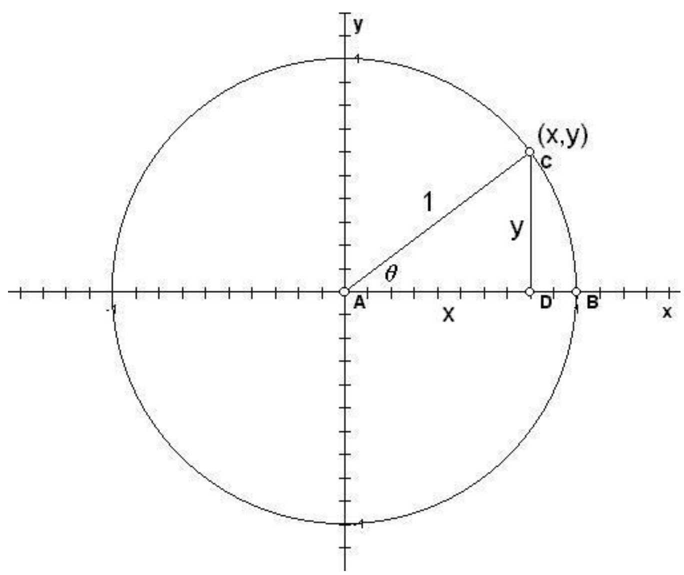
Of course, all of the other angles between $0°$ and $360°$ have corresponding radian measures, they just are not as nice as the ones above. For example, the angle $29°$ corresponds to $29 \cdot \frac{\pi}{180} = \frac{29\pi}{180}$ radians. Since this is not a particularly nice angel we tend to convert it to decimal form and simply say that it is 0.506145483078 radians. We will find that the decimal representation comes in handy when we shift over to the function representation of the trigonometric functions.

Before we move to the function representation let's continue with the unit circle. We have two major things to do. First we need to get a correspondence between the (x, y) points on the unit circle and the trigonometry of the triangle. Second we need to use symmetry to find the trigonometric functions for each of the nice angles above.

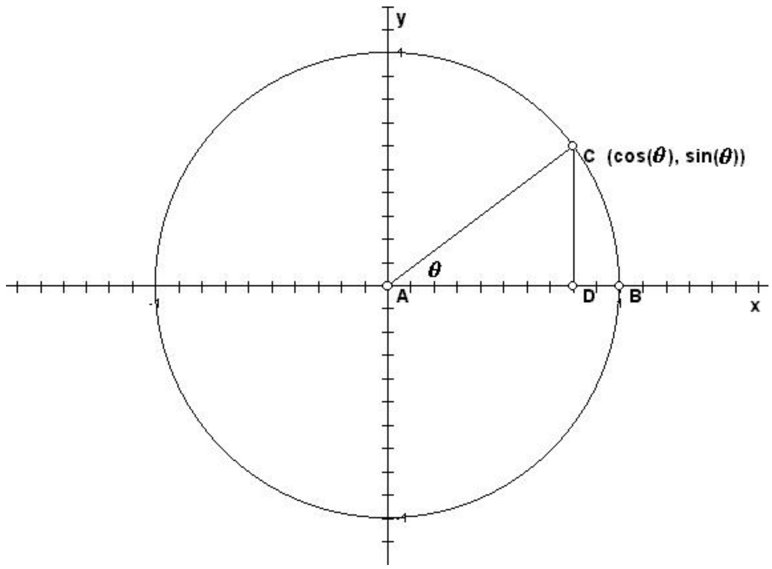
The correspondence between the (x, y) points on the unit circle and the trigonometry of the triangle is probably the most important relationship in trigonometry. To get this relationship we will place a right triangle inside the unit circle so that its hypotenuse is a radius. We will call the angle between the positive x axis and the hypotenuse θ and our trigonometric functions will be of the angle θ.



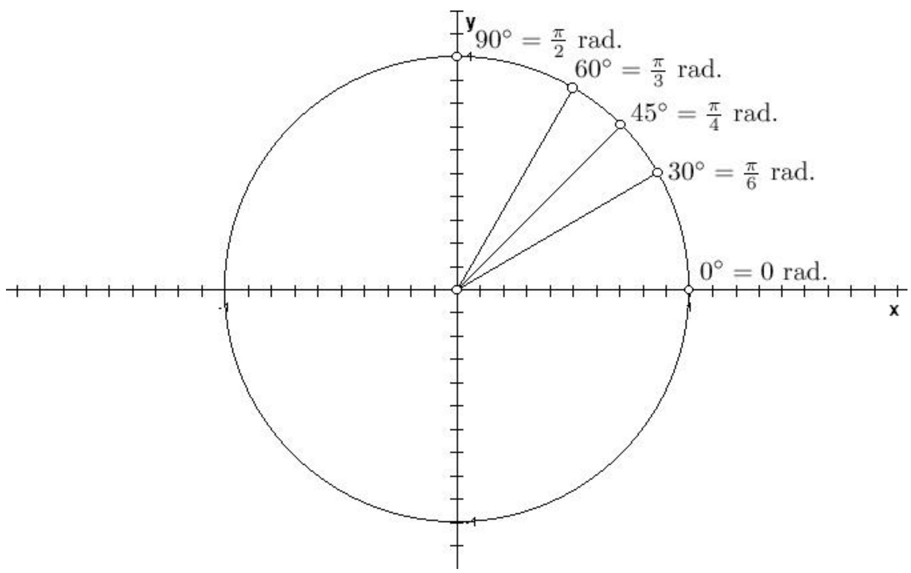
We will also denote the coordinates of the point C as (x, y) and our charge is to find the relationship between the x and y values of that point and the trigonometric functions of θ. The key is the inside triangle. First note that the lengths of the legs of the triangle are x and y.



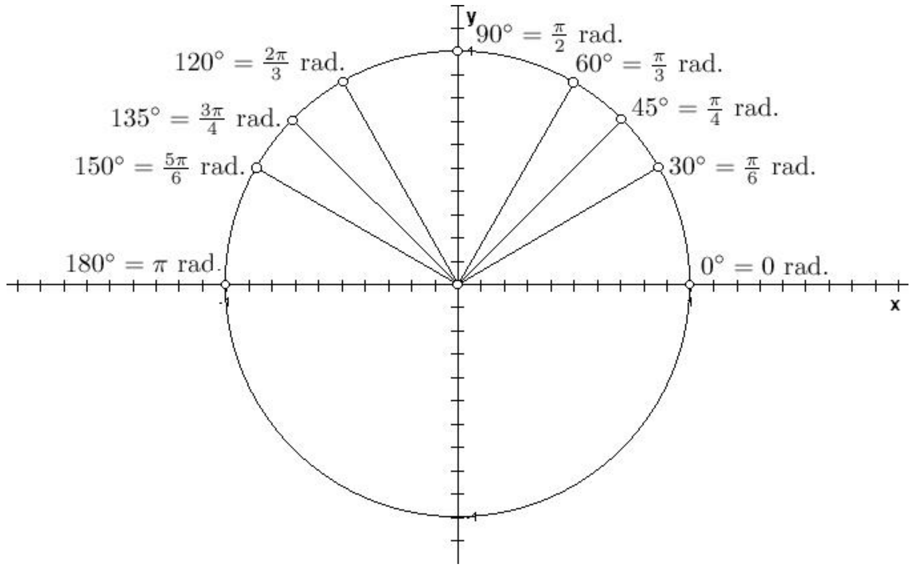
Now consider only the triangle inside the circle. The $\sin(\theta) = \frac{opp}{hyp} = \frac{y}{1} = y$ and $\cos(\theta) = \frac{adj}{hyp} = \frac{x}{1} = x$. So we have,



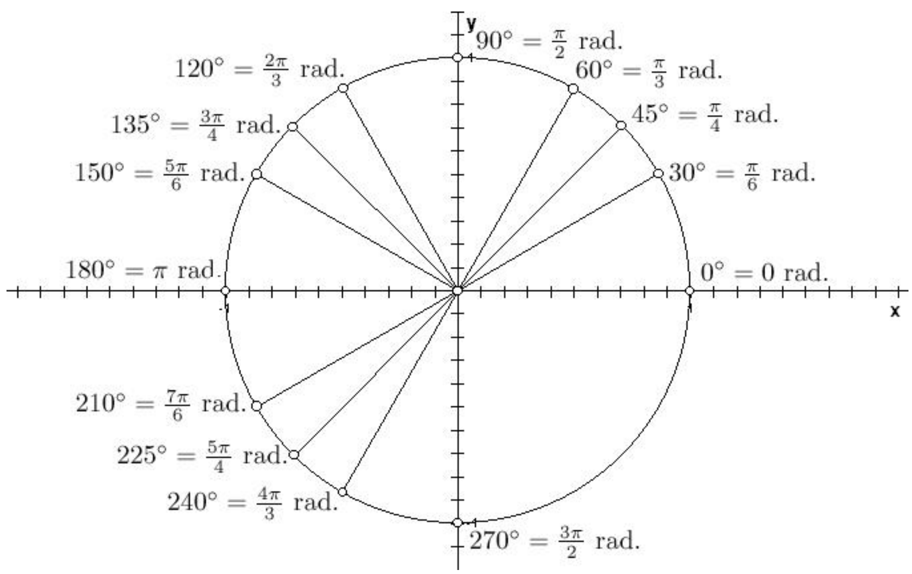
So every point on the unit circle has the coordinates $(\cos(\theta), \sin(\theta))$. Now if we put this observation together with some symmetry we can obtain the trigonometric functions at the nice angles. First, let's place all of the nice angles on the unit circle. We will start with the three nice angles from the triangle representation (30°, 45° and 60°) along with 0° and 90°. These angles correspond to 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ radians respectively.



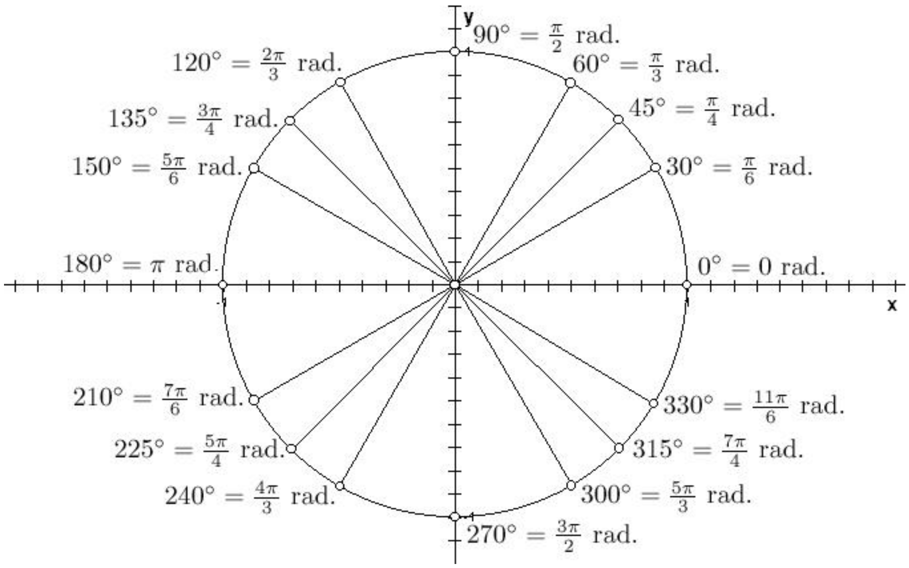
This takes care of the first quadrant. In the second quadrant, we take the nice angles (30°, 45°, 60° and 90°) and add 90° to each. Similarly, in radian measure we take the nice angles ($\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$) and add $\frac{\pi}{2}$ to each.



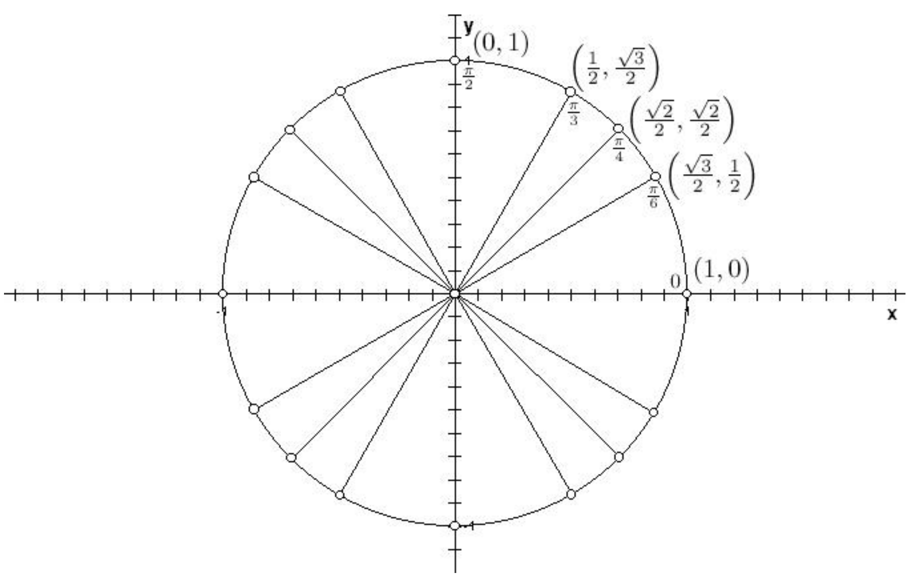
This takes care of the second quadrant. In the third quadrant, we can think of the nice angles in a couple ways. First, we could take the nice angles from the first quadrant (30°, 45°, 60° and 90°) and add 180° to each. Similarly, in radian measure we would take ($\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$) and add π to each. Another way to think about the nice angles in the third quadrant is to take the nice angles in the second quadrant (120°, 135°, 150° and 180°) and add 90° to each. Similarly, in radian measure we would take ($\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{5\pi}{6}$ and π) and add $\frac{\pi}{2}$ to each. In either case we get the angles, 210°, 225°, 240° and 270°, which corresponds to $\frac{7\pi}{6}$, $\frac{5\pi}{4}$, $\frac{4\pi}{3}$ and $\frac{3\pi}{2}$ respectively.



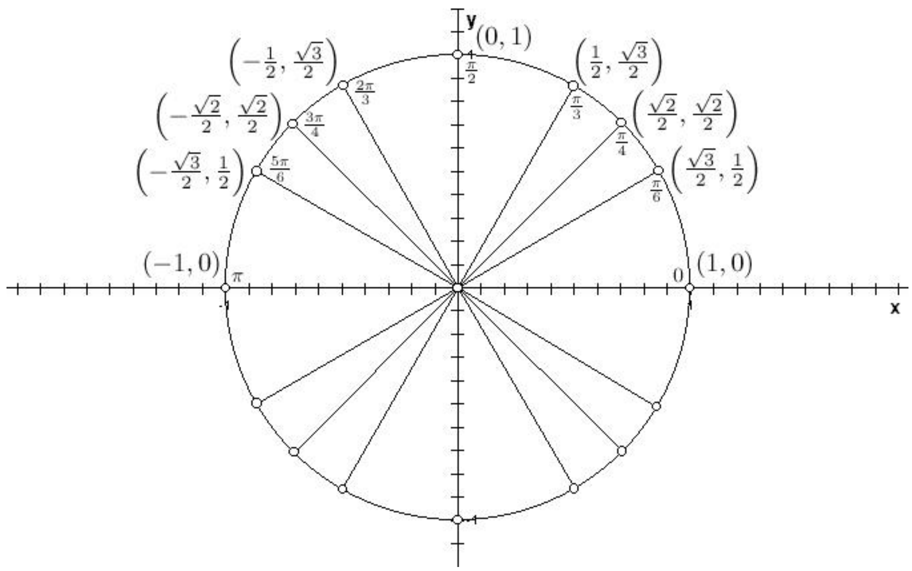
Finally, in the fourth quadrant, we can construct the nice angles in three different ways from the angles we already have. First, we could take the nice angles from the first quadrant and add 270° ($\frac{3\pi}{2}$ radians). Second, we could take the nice angles from the second quadrant and add 180° (π radians). Third, we could take the nice angles from the third quadrant and add 90° ($\frac{\pi}{2}$ radians). In any case we get the angles, 300°, 315°, 330° and 360°, which corresponds to $\frac{5\pi}{3}$, $\frac{7\pi}{4}$, $\frac{11\pi}{6}$ and 2π respectively.



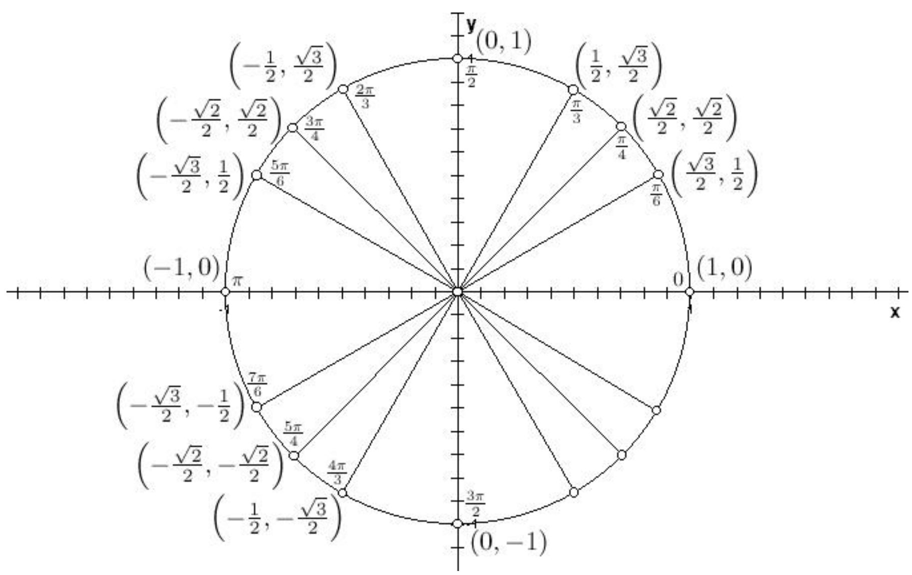
Now we will construct the unit circle "table" that will give us all of the trigonometric function values at each of the nice angles. To do this we need only find the coordinates to the points on the unit circle at each of the angles. From our above discussion we know that these points are of the form $(\cos(\theta), \sin(\theta))$. So we will be able to simply read off the sine and cosine values and the other four will be easily derived through a ratio of the values. Also from the above discussion, we know the sine and cosine values for the angles $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$. Adding these to the angles 0 and $\frac{\pi}{2}$, which are trivial to determine, we have.



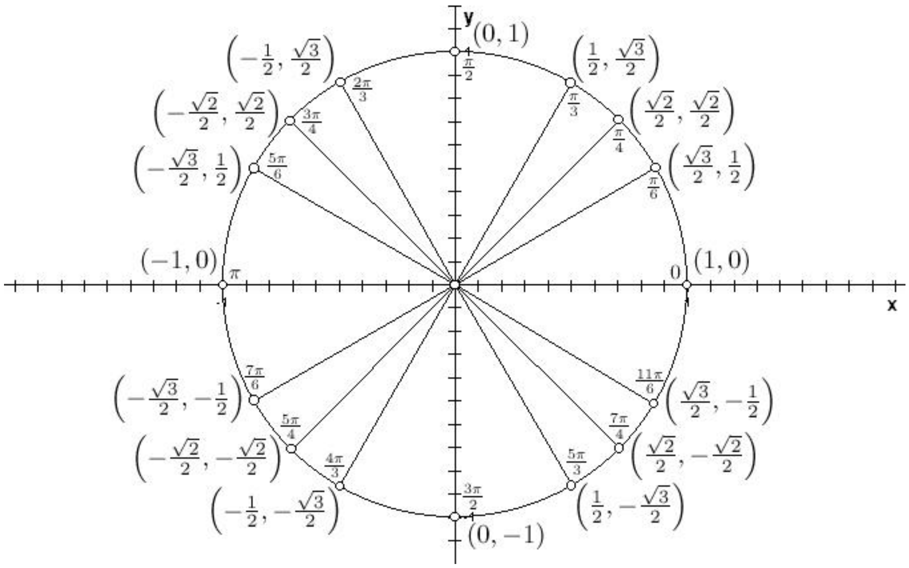
The points in the other three quadrants are easy to determine through symmetry. For example, the second quadrant can be obtained from the first by reflecting the points about the y-axis. When you reflect the point (x, y) about the y-axis you get the point $(-x, y)$. So to fill in the chart for the second quadrant we simply change the signs of the x coordinates.



To get the third quadrant values we could either reflect the second quadrant points about the x-axis (effectively changing all of the signs of the y coordinates) or by reflecting the first quadrants points about the origin (changing the signs of both coordinates).



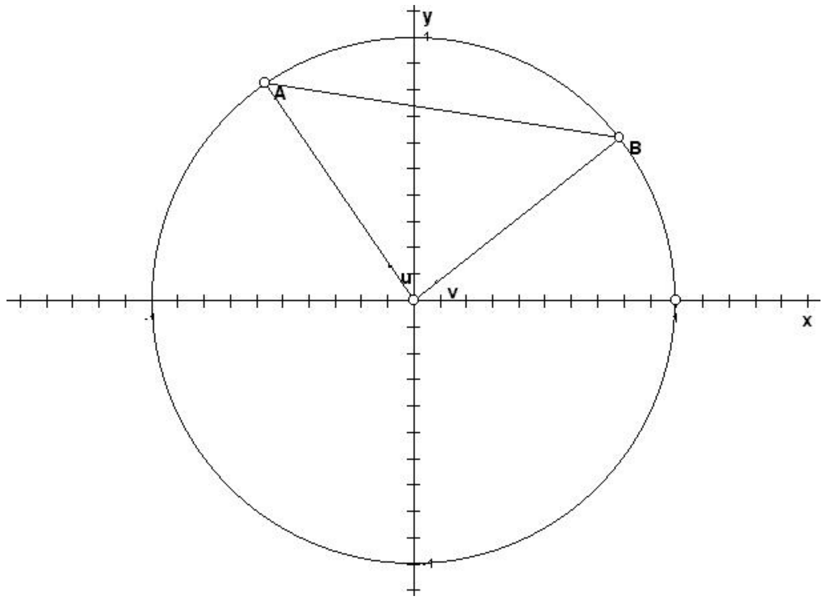
Finally, to finish it all up we simply need to fill in the fourth quadrant. This can be done in any one of three ways. We could reflect the first quadrant over the x-axis or reflect the third quadrant over the y-axis or reflect the second quadrant over the origin.



That completes the unit circle chart of the trigonometric functions of the nice angles. Memorizing the entire chart is not necessary if one simply remembers the way we constructed the chart from just the three points in the first quadrant. In fact, this is overkill as well. To construct the chart you need only remember three values $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$ and the fact that $\frac{1}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$. Then the chart can be constructed by imagining where the angle lies and then placing the corresponding values in the x and y coordinates, finally put the signs on the coordinates that correspond with the quadrant the point is in.

1.3.2 More Trigonometric Identities

With unit circle trigonometry we can derive many more useful trigonometric identities. We will look at only a few of the more important ones here. The identities that spawn most of the others are the formulas for the sine or cosine of angle sums and differences. We will derive those now. Consider the following diagram where A and B are two points on the unit circle at angles u and v respectively.



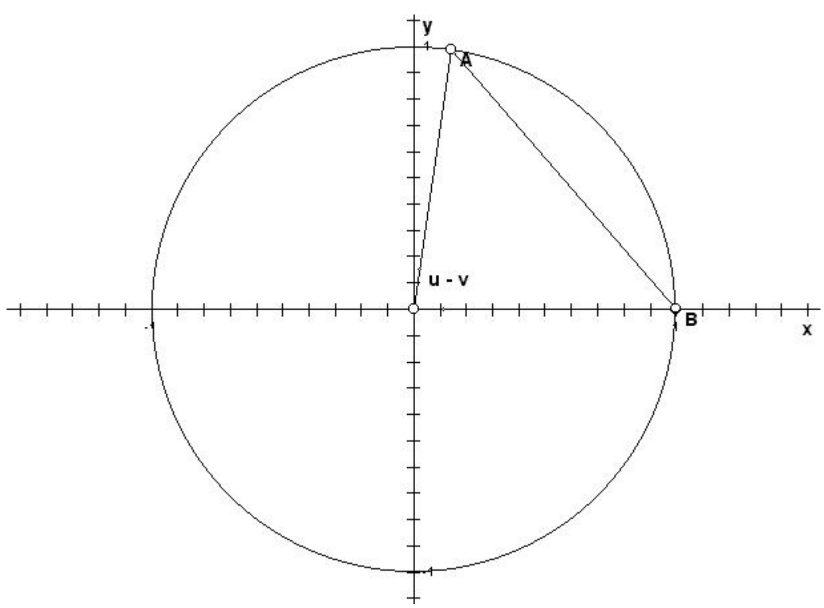
Using the distance formula we have that the distance between A and B is

$$AB = \sqrt{(\cos(u) - \cos(v))^2 + (\sin(u) - \sin(v))^2}$$

Doing a little algebra we have,

$$\begin{aligned} AB &= \sqrt{(\cos(u) - \cos(v))^2 + (\sin(u) - \sin(v))^2} \\ &= \sqrt{\cos^2(u) - 2\cos(u)\cos(v) + \cos^2(v) + \sin^2(u) - 2\sin(u)\sin(v) + \sin^2(v)} \\ &= \sqrt{2 - 2(\cos(u)\cos(v) + \sin(u)\sin(v))} \end{aligned}$$

Now we will rotate the image clockwise so that B lies on the x-axis.



Note that since we rotated clockwise by an angle of v the angle to the point A is now $u - v$. If we take the distance between A and B, noting that B is at $(1, 0)$, we get.

$$AB = \sqrt{(\cos(u - v) - 1)^2 + (\sin(u - v) - 0)^2}$$

Doing a little algebra we have,

$$\begin{aligned} AB &= \sqrt{(\cos(u - v) - 1)^2 + (\sin(u - v) - 0)^2} \\ &= \sqrt{\cos^2(u - v) - 2\cos(u - v) + 1 + \sin^2(u - v)} \\ &= \sqrt{2 - 2\cos(u - v)} \end{aligned}$$

Setting these two equations equal to each other gives,

$$\begin{aligned} \sqrt{2 - 2\cos(u - v)} &= \sqrt{2 - 2(\cos(u)\cos(v) + \sin(u)\sin(v))} \\ 2 - 2\cos(u - v) &= 2 - 2(\cos(u)\cos(v) + \sin(u)\sin(v)) \\ -2\cos(u - v) &= -2(\cos(u)\cos(v) + \sin(u)\sin(v)) \\ \cos(u - v) &= \cos(u)\cos(v) + \sin(u)\sin(v) \end{aligned}$$

Which is our first difference angle formula. Using the fact that $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$, both of which can be seen from the unit circle constructions above.

$$\begin{aligned} \cos(u + v) &= \cos(u - (-v)) \\ &= \cos(u)\cos(-v) + \sin(u)\sin(-v) \\ &= \cos(u)\cos(v) - \sin(u)\sin(v) \end{aligned}$$

Also, we can get the first of our double angle formulas as

$$\begin{aligned} \cos(2u) &= \cos(u + u) \\ &= \cos(u)\cos(u) - \sin(u)\sin(u) \\ &= \cos^2(u) - \sin^2(u) \end{aligned}$$

Let's move onto the sine by converting the cosine formulas into sine formulas. Note that

$$\begin{aligned} \cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta) \\ &= \sin(\theta) \end{aligned}$$

and thus

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right) \\ &= \cos(\theta) \end{aligned}$$

So,

$$\begin{aligned} \sin(u+v) &= \cos\left(\frac{\pi}{2}-(u+v)\right) \\ &= \cos\left(\left(\frac{\pi}{2}-u\right)-v\right) \\ &= \cos\left(\frac{\pi}{2}-u\right)\cos(v)+\sin\left(\frac{\pi}{2}-u\right)\sin(v) \\ &= \sin(u)\cos(v)+\cos(u)\sin(v) \end{aligned}$$

and

$$\begin{aligned} \sin(u-v) &= \sin(u)\cos(-v)+\cos(u)\sin(-v) \\ &= \sin(u)\cos(v)-\cos(u)\sin(v) \end{aligned}$$

Which also gives a double angle formula of,

$$\begin{aligned} \sin(2u) &= \sin(u+u) \\ &= \sin(u)\cos(u)+\cos(u)\sin(u) \\ &= 2\sin(u)\cos(u) \end{aligned}$$

We can use these to construct formulas for the other trig functions as well. For example,

$$
\begin{aligned}
\tan(u+v) &= \frac{\sin(u+v)}{\cos(u+v)} \\
&= \frac{\sin(u)\cos(v)+\cos(u)\sin(v)}{\cos(u)\cos(v)-\sin(u)\sin(v)} \\
&= \frac{\sin(u)\cos(v)+\cos(u)\sin(v)}{\cos(u)\cos(v)-\sin(u)\sin(v)} \cdot \frac{\frac{1}{\cos(u)\cos(v)}}{\frac{1}{\cos(u)\cos(v)}} \\
&= \frac{\tan(u)+\tan(v)}{1-\tan(u)\tan(v)}
\end{aligned}
$$

and

$$
\begin{aligned}
\tan(u-v) &= \frac{\sin(u-v)}{\cos(u-v)} \\
&= \frac{\sin(u)\cos(v)-\cos(u)\sin(v)}{\cos(u)\cos(v)+\sin(u)\sin(v)} \\
&= \frac{\sin(u)\cos(v)-\cos(u)\sin(v)}{\cos(u)\cos(v)+\sin(u)\sin(v)} \cdot \frac{\frac{1}{\cos(u)\cos(v)}}{\frac{1}{\cos(u)\cos(v)}} \\
&= \frac{\tan(u)-\tan(v)}{1+\tan(u)\tan(v)}
\end{aligned}
$$

Two more that come in handy are called the half angle formulas since they work well in converting a trigonometric function of an angle to a function of twice that angle.

$$
\begin{aligned}
\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) \\
\cos(2\theta)+1 &= \cos^2(\theta)-\sin^2(\theta)+\sin^2(\theta)+\cos^2(\theta) \\
\cos(2\theta)+1 &= 2\cos^2(\theta) \\
\cos^2(\theta) &= \frac{\cos(2\theta)+1}{2} \\
\cos(\theta) &= \sqrt{\frac{\cos(2\theta)+1}{2}}
\end{aligned}
$$

Keep in mind that this last formula is true in magnitude only. If θ is such that $\cos(\theta) < 0$ then the formula is $\cos(\theta) = -\sqrt{\frac{\cos(2\theta)+1}{2}}$. As for the sine,

$$
\begin{aligned}
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
\cos(2\theta) - 1 &= \cos^2(\theta) - \sin^2(\theta) - (\sin^2(\theta) + \cos^2(\theta)) \\
\cos(2\theta) - 1 &= -2\sin^2(\theta) \\
\sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \\
\sin(\theta) &= \sqrt{\frac{1 - \cos(2\theta)}{2}}
\end{aligned}
$$

Again this last formula is true in magnitude only. If θ is such that $\sin(\theta) < 0$ then the formula is $\sin(\theta) = -\sqrt{\frac{1-\cos(2\theta)}{2}}$.

In summary,

$$
\begin{aligned}
\cos(u - v) &= \cos(u)\cos(v) + \sin(u)\sin(v) \\
\cos(u + v) &= \cos(u)\cos(v) - \sin(u)\sin(v) \\
\cos(2u) &= \cos^2(u) - \sin^2(u) \\
\cos\left(\frac{\pi}{2} - \theta\right) &= \sin(\theta) \\
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos(\theta) \\
\sin(u + v) &= \sin(u)\cos(v) + \cos(u)\sin(v) \\
\sin(u - v) &= \sin(u)\cos(v) - \cos(u)\sin(v) \\
\sin(2u) &= 2\sin(u)\cos(u) \\
\tan(u + v) &= \frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)} \\
\tan(u - v) &= \frac{\tan(u) - \tan(v)}{1 + \tan(u)\tan(v)} \\
\cos^2(\theta) &= \frac{\cos(2\theta) + 1}{2} \\
\cos(\theta) &= \pm\sqrt{\frac{\cos(2\theta) + 1}{2}} \\
\sin^2(\theta) &= \frac{1 - \cos(2\theta)}{2} \\
\sin(\theta) &= \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}
\end{aligned}
$$

Of course, we could go on and on with these but this is probably enough. Trigonometric identities come in handy when simplifying trigonometric expressions, which is sometimes rather difficult. We can also use them to find the exact values of trigonometric functions of angles that are not the nice angles. For example,

Example 1.3.1 :

$$\sin(15^\circ) = \sqrt{\frac{1-\cos(30^\circ)}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}$$

But the real use is in the manipulation of trigonometric expressions. In Calculus, especially when doing integration, changing the form of the trigonometric expression may make the difference between an easy problem and one that is nearly impossible.

1.4 Functions

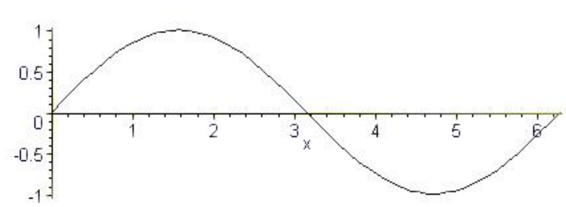
The third way to view trigonometric functions is as functions graphed on the cartesian coordinate system. To construct these functions we take the unit circle and we let the angle θ be our independent variable, so it is associated with x. We let our dependent variable (y) be the trigonometric function of that angle. So for example, since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ is on the cosine curve. So for the sine curve,

θ	$\sin(\theta)$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

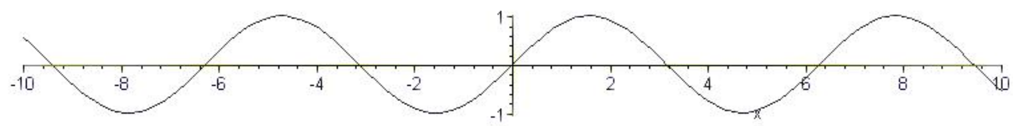
$\Longrightarrow$

x	y
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

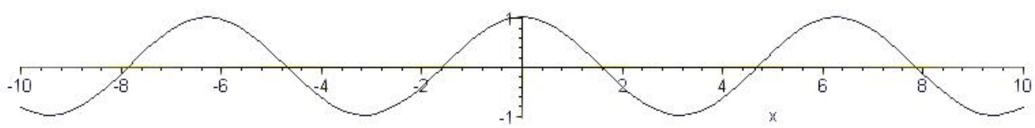
Graphing these values along with the other angles between 0 and 2π gives,



Since the next set of values are simply these plus another 2π and the circle, is well a circle, the values will repeat themselves. In fact they will repeat themselves indefinitely in both directions. So the sine function will look like,

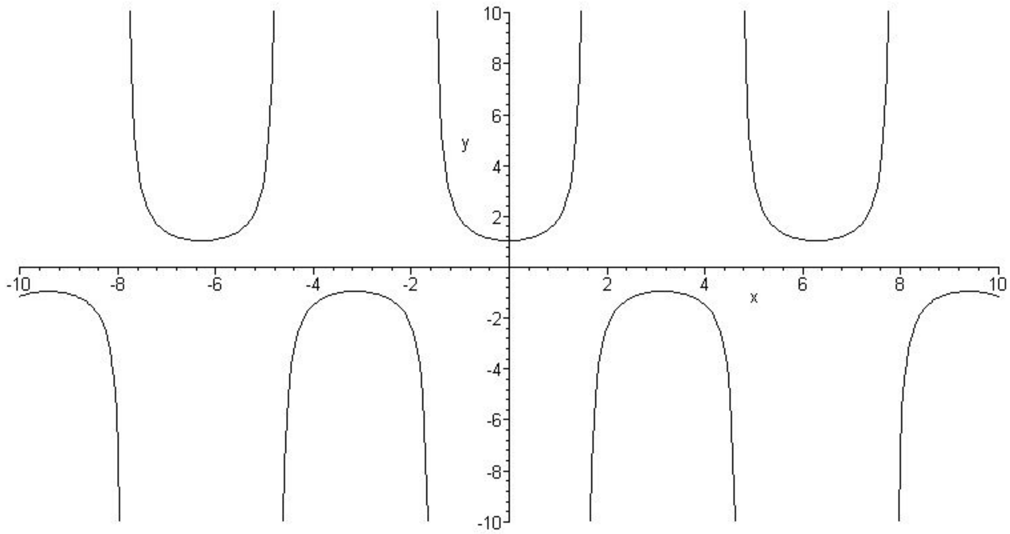


We construct the cosine function in the same manner.

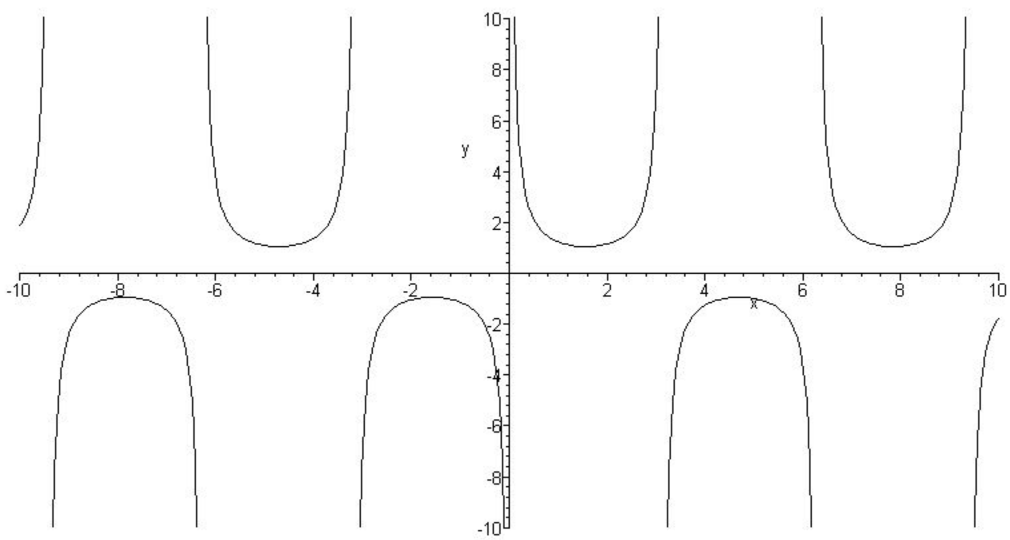


Trigonometric functions are a subset of a much larger class of functions called **Periodic Functions**. Periodic functions are functions that repeat themselves. As we saw above both the sine and cosine repeat themselves and hence they are periodic functions. The **Period** of a periodic function is width of the smallest portion of the graph that does repeat itself. For the sine function there are many portions that repeat themselves. For example, the portion from 0 to 4π, the portion from 0 to 6π, the portion from 0 to 2π, the portion from -2π to 6π, the portion from -2π to 2π, the portion from -2π to 0, and so on. You will notice that the smallest portions that repeat themselves are all 2π in length. So the period of the sine function is 2π. The period for the cosine function is also 2π.

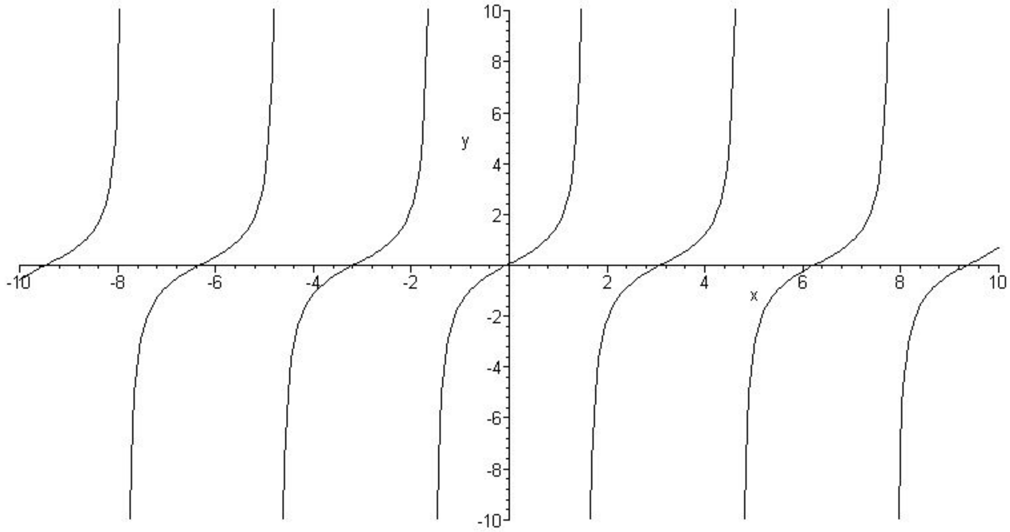
Since the secant and cosecant functions are reciprocals of the cosine and sine functions respectively their periods will also be 2π. The secant function,



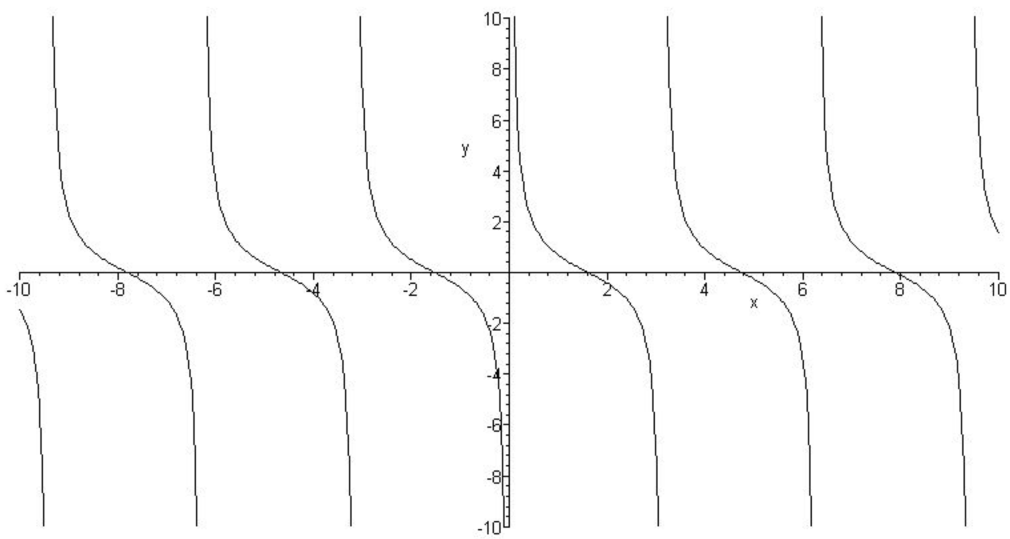
and the cosecant function,



Since the tangent and cotangent functions are ratios of sine and cosine it turns out that their periods are shorter, in fact exactly half of the period of sine and cosine, π. The tangent function,

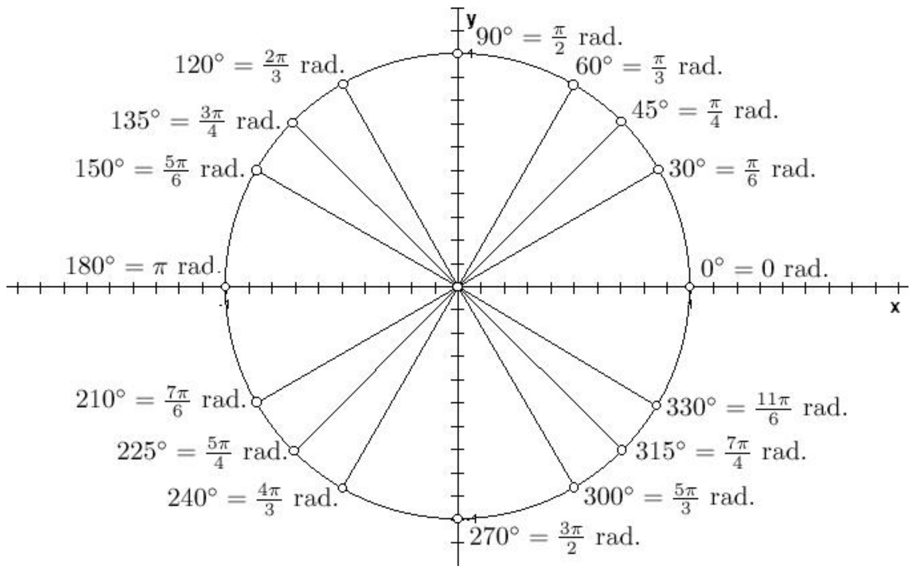


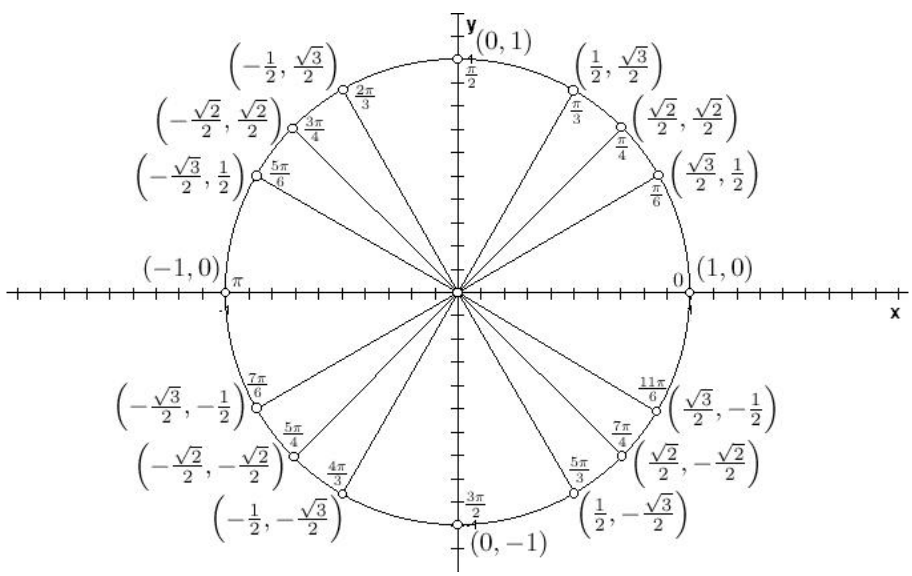
and the cotangent function,



1.5 Quick Reference

Unit Circle Diagrams





Identity Summary

$$
\begin{aligned}
\sin^2(\theta)+\cos^2(\theta) &= 1\\
\sin^2(\theta) &= 1-\cos^2(\theta)\\
\cos^2(\theta) &= 1-\sin^2(\theta)\\
\sec^2(\theta) &= \tan^2(\theta)+1\\
\csc^2(\theta) &= \cot^2(\theta)+1\\
\sin(-\theta) &= -\sin(\theta)\\
\cos(-\theta) &= \cos(\theta)\\
\cos(u-v) &= \cos(u)\cos(v)+\sin(u)\sin(v)\\
\cos(u+v) &= \cos(u)\cos(v)-\sin(u)\sin(v)\\
\cos(2u) &= \cos^2(u)-\sin^2(u)\\
\cos(2u) &= 2\cos^2(u)-1\\
\cos(2u) &= 1-2\sin^2(u)\\
\cos\left(\frac{\pi}{2}-\theta\right) &= \sin(\theta)\\
\sin\left(\frac{\pi}{2}-\theta\right) &= \cos(\theta)\\
\sin(u+v) &= \sin(u)\cos(v)+\cos(u)\sin(v)\\
\sin(u-v) &= \sin(u)\cos(v)-\cos(u)\sin(v)\\
\sin(2u) &= 2\sin(u)\cos(u)\\
\tan(u+v) &= \frac{\tan(u)+\tan(v)}{1-\tan(u)\tan(v)}\\
\tan(u-v) &= \frac{\tan(u)-\tan(v)}{1+\tan(u)\tan(v)}\\
\cos^2(\theta) &= \frac{\cos(2\theta)+1}{2}\\
\cos(\theta) &= \pm\sqrt{\frac{\cos(2\theta)+1}{2}}\\
\sin^2(\theta) &= \frac{1-\cos(2\theta)}{2}\\
\sin(\theta) &= \pm\sqrt{\frac{1-\cos(2\theta)}{2}}\\
\sin(u)\cos(v) &= \frac{1}{2}\left[\sin(u+v)+\sin(u-v)\right]\\
\cos(u)\cos(v) &= \frac{1}{2}\left[\cos(u+v)+\cos(u-v)\right]\\
\sin(u)\sin(v) &= \frac{1}{2}\left[\cos(u-v)-\cos(u+v)\right]
\end{aligned}
$$