Cylinder Reflections
The Mathematics Behind the Images

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Anamorphic art is created by distorting an image so that is only revealed from a single vantage point or from its reflection on a mirrored surface. This artistic process was first attempted during the Renaissance and became exceedingly popular during the Victorian Era.

The earliest known examples come from the notebooks of Leonardo da Vinci. He successfully sketched an eyeball in 1485 that could only be discerned when looking at the drawing from a certain angle.
Anamorphic Art

More modern artists using these techniques include Julian Beever, who creates three-dimensional illusions on sidewalks using chalk.

Julian Beever’s Fountain
Anamorphic Art

Julian Beever’s Fishing
Anamorphic Art

Julian Beever’s Rafting

As you can see, Julian Beever likes to incorporate human subjects in all of his sidewalk art. This shows that not only can he create the perspective shift but he can do it to scale.
Anamorphic Art

Hans Hamngren and István Orosz use the mirrored cylinder technique. They achieve this illusion by either drawing the image on a distorted grid, similar to the way M. C. Escher created many of his illusions, or looking at the mirrored image while drawing on a flat surface.

Pictured to the left is the work of István Orosz.
Anamorphic Art

More of István Orosz’s work.
Anamorphic Art

This is a work by Hans Hamngren, an old fire hydrant in an old fire extinguisher.
The Process

Step 1: The Setup — Cylinder
Step 1: The Setup — Paper
The Process

Step 1: The Setup — Viewer Position
Step 1: The Setup — The Final Image
Step 1: The Setup — Place the image in the cylinder.
The Process

Step 2: Draw a line from pixel to viewer.
The Process

Step 3: Find the intersection with cylinder.
Step 3: Find the intersection with cylinder.

Step 3: Consider the line from intersection to viewer.
The Process

Step 4: Find normal line from intersection.
Step 5: Find the reflection line from intersection.
Step 6: Find the intersection of reflection line and paper.
Step 6: Plot the point.
Step 6: Repeat for all pixels on the image.
Step 1: The Setup

Assumptions: The base of the cylinder is on the $xy$-plane, the central axis passes through the origin, and the paper is on the $xy$-plane.

\[
\mathbf{P} = \langle p_x, p_y, p_z \rangle \\
\mathbf{V} = \langle v_x, v_y, v_z \rangle \\
r^2 = x^2 + y^2
\]
Step 2: Draw a line from pixel to viewer.

To do this we take the starting position to be the pixel point $\mathbf{P}$ and the direction to be toward the viewer $\mathbf{V} - \mathbf{P}$. The corresponding formulas are,

$$L(t) = \mathbf{P} + t(\mathbf{V} - \mathbf{P})$$

$$= \langle p_x, p_y, p_z \rangle + t(\langle v_x, v_y, v_z \rangle - \langle p_x, p_y, p_z \rangle)$$

$$= \langle p_x + t(v_x - p_x), p_y + t(v_y - p_y), p_z + t(v_z - p_z) \rangle$$

As $t$ goes from 0 to 1 we trace out the line segment from $\mathbf{P}$ to $\mathbf{V}$. 

\[ \]
Step 3: Find the intersection with cylinder.

To do this we take the line from the pixel to the viewer and plug the $x$ and $y$ expressions into the equation of the cylinder and solve for $t$.

$$r^2 = x^2 + y^2$$
$$= \left( p_x + t(v_x - p_x) \right)^2 + \left( p_y + t(v_y - p_y) \right)^2$$
$$= p_x^2 + p_y^2 + t(-2p_x^2 - 2p_y^2 + 2p_x v_x + 2p_y v_y)$$
$$+ t^2(p_x^2 + p_y^2 - 2p_x v_x + v_x^2 - 2p_y v_y + v_y^2)$$
$$= at^2 + bt + c$$
Step 3: Find the intersection with cylinder.

Now solve the quadratic equation,

\[ at^2 + bt + c - r^2 = 0 \]

for \( t \) and we get,

\[ t = \frac{-b \pm \sqrt{b^2 - 4a(c - r^2)}}{2a} \]

The one we want is

\[ t_i = \frac{-b + \sqrt{b^2 - 4a(c - r^2)}}{2a} \]
Step 3: Find the intersection with cylinder.

Now we substitute this value, \( t_i \), in for \( t \) in the line equations to get the point of intersection.

\[
L(t_i) = P + t_i(V - P)
\]
\[
= \langle p_x + t_i(v_x - p_x), p_y + t_i(v_y - p_y), p_z + t_i(v_z - p_z) \rangle
\]
\[
= \langle l_x, l_y, l_z \rangle
\]
Step 4: Find the normal vector from intersection.

The normal vector is perpendicular to the surface and is used in the calculation of the reflection line. For a cylinder, the normal vector will be parallel to the $xy$-plane and pass through the points $\langle I_x, I_y, I_z \rangle$ and $\langle 0, 0, I_z \rangle$. So our normal vector is the difference between these two points,

$$\mathbf{n} = \langle I_x, I_y, 0 \rangle$$
Step 5: Find the reflection vector from intersection.

This is probably the most involved calculation in the process. From the diagram on the right notice that the reflection vector \( \mathbf{r} = \mathbf{v} + 2\mathbf{a} \) where \( \mathbf{v} = \mathbf{V} - \mathbf{P} \) is the vector from the pixel to the viewer.

Since

\[ \text{Proj} = \mathbf{v} + \mathbf{a} \]

we have

\[ \mathbf{a} = \text{Proj} - \mathbf{v} \]
Step 5: Find the reflection vector from intersection.

\[ \text{Proj} = \frac{n \cdot v}{|n|^2} n \]

So

\[ a = \frac{n \cdot v}{|n|^2} n - v \]
Step 5: Find the reflection vector from intersection.

\[ r = v + 2a \]
\[ = v + 2 \left( \frac{n \cdot v}{|n|^2} n - v \right) \]
\[ = \frac{2n \cdot v}{|n|^2} n - v \]
\[ = \langle r_x, r_y, r_z \rangle \]
Step 5: Find the reflection line from intersection.

So our reflection line is

\[ R(t) = \langle l_x, l_y, l_z \rangle + t \langle r_x, r_y, r_z \rangle \]
\[ = \langle l_x + tr_x, l_y + tr_y, l_z + tr_z \rangle \]
Step 6: Find the intersection of reflection line and paper.

The paper is on the $xy$-plane so every three dimensional point on the paper has a $z$ coordinate of 0. We can use this fact to find how far we must move down the reflection vector until we hit the paper, this is the value of $t$ in the reflection line formula,

$$R(t) = \langle l_x + tr_x, l_y + tr_y, l_z + tr_z \rangle$$
Step 6: Find the intersection of reflection line and paper.

If we set the $z$ coordinate of the reflection line equal to 0 we can solve for $t$,

$$l_z + tr_z = 0$$

gives

$$t = -\frac{l_z}{r_z}$$
Step 6: Find the intersection of reflection line and paper.

Substitute this value in for $t$ in the reflection line equation gives the intersection point

$$\left< l_x - \frac{l_z}{r_z} r_x, l_y - \frac{l_z}{r_z} r_y, l_z - \frac{l_z}{r_z} r_z \right>$$

which gives

$$\left< l_x - \frac{l_z}{r_z} r_x, l_y - \frac{l_z}{r_z} r_y, 0 \right>$$
Step 6: Plot the Point

Plot this point on the paper in the same color as the original pixel color and move on to the next pixel. When all of the points are plotted you have your transformed image.
Who did the work?

The calculations for this and several other anamorphic scenario were done as undergraduate research projects by students at Salisbury University.

- Nicole Massarelli (2010) — Theoretical extensions to general convex surfaces. She also wrote the software for cylinder reflections.
- Angela Rose and Erika Gerhold (2011) — Tilted Cylinder