Chapter 5 – Discounted Cash Flow Valuation

Compounding Periods Other Than Annual

Let’s examine monthly compounding problems.

Future Value

Suppose you invest $9,000 today and get an interest rate of 9 percent compounded monthly. How much will you have in 3 years?

\[ FV = \]

\[ \text{N} \]
\[ \text{I} \]
\[ \text{PV} \]
\[ \text{Pmt} \]
\[ \text{Cpt FV} \] $11,777.81

You can see I have introduced another variable. This variable, \( m \), is the number of compounding periods per year. If it is monthly compounding, then \( m \) is equal to 12. If it is quarterly compounding, then \( m \) is equal to 4. The number of years is multiplied by \( m \) and the annual percentage rate (APR) is divided by \( m \).

Present Value

Assume you need $500,000 in 30 years and can earn an interest rate of 6 percent compounded monthly. How much will you have to invest today?

\[ PV = \]

\[ \text{N} \]
\[ \text{I} \]
\[ \text{Cpt PV} \] $83,020.96
\[ \text{Pmt} \]
\[ \text{FV} \]
Interest Rate
You have $10,000 and need $12,500 in 2.5 years. What interest rate must you earn to accomplish your goal? Assume monthly compounding.

\[
\begin{align*}
N & \quad \text{Cpt} I \quad 0.7466\% = r/12 \rightarrow r = (0.7466)(12) = 8.959\% \\
PV & \\
Pmt & \\
FV &
\end{align*}
\]

Number of Periods
You have $100,000 and need $125,000. Your investment will earn a rate of 15.75 \% compounded monthly. How many years before you accomplish your goal?

\[
\begin{align*}
\text{Cpt} N & \quad 17.11 = t(12) \rightarrow t=17.11/12 = 1.43 \text{ years} \\
I & \\
PV & \\
Pmt & \\
FV &
\end{align*}
\]

This method for calculating the different variables in the lump sum equations can be applied to uneven, perpetuity, and annuity cash flows.
Uneven Cash Flows

Suppose you are going to receive $10,000 in one year, $15,000 in two years and three years, and $20,000 in five years. If the appropriate interest rate is 10 percent, what is the value of the cash flows today? Draw the time line.

\[
PV = 10,000(PVIF_{10\%}^1) + 15,000(PVIF_{10\%}^2) + 15,000(PVIF_{10\%}^3) + 20,000(PVIF_{10\%}^5)
\]

\[
N = N = N = N = \quad I = I = I = I = \quad N = N = N = N = \quad I = I = I = I =
\]

\[
\begin{array}{l|c}
\text{Cpt PV} & 9,090.91 \\
\text{FV} = & \quad \text{Cpt PV} = 12,396.69 \\
\text{Pmt} = & \quad \text{Cpt PV} = 11,269.72 \\
\text{PV} = & \quad \text{Cpt PV} = 12,418.43 \\
\end{array}
\]

\[
PV_0 = 9,090.91 + 12,396.69 + 11,269.72 + 12,418.43 = 45,175.75
\]

What if I want to know the value at year 5?

\[
FV_5 = 10,000(FVIF_{10\%}^4) + 15,000(FVIF_{10\%}^3) + 15,000(FVIF_{10\%}^2) + 20,000
\]

\[
N = N = N = N = \quad I = I = I = I = \quad PV = PV = PV = PV = \\
\begin{array}{l|c}
\text{Cpt FV} = 14,641.00 \\
\text{FV} = \quad \text{Cpt FV} = 19,965.00 \\
\text{Pmt} = \quad \text{Cpt FV} = 18,150.00 \\
\text{PV} = \quad \text{Cpt FV} = 20,000 \\
\end{array}
\]

\[
V_5 = 14,641.00 + 19,965.00 + 18,150.00 + 20,000.00 = 72,756.00
\]
Another approach is:

\[ FV = PV(FVIF_{r,t}) \rightarrow FV = $45,175.75(FVIF_{10\%,5}) \]

\[ \begin{align*}
N &= \\
I &= \\
PV &= \\
Pmt &= \\
\text{Cpt FV} &= 72,756.00
\end{align*} \]

Cash Flow Worksheet

The cash flow worksheet is another alternative to solving this problem. The cash flow worksheet is initiated with the CF key. Push this key and you see \( CF_0 \).

In this example, the cash flows are:

| \( CF_0 \) | 0 |
| \( CF_1 \) | 10,000 | \( F_1 \) | 1 |
| \( CF_2 \) | 15,000 | \( F_2 \) | 2 |
| \( CF_3 \) | 0 | \( F_3 \) | 1 |
| \( CF_4 \) | 20,000 | \( F_4 \) | 1 |

\[ \begin{align*}
I &= 10 \\
\text{Cpt NPV} &= 45,175.75
\end{align*} \]

Perpetuities

A perpetuity is an even cash flow that occurs at even time intervals forever.

You will receive $75,000 per year forever with the first payment occurring one year from today. If the interest rate is 6 percent, what is the value of the perpetuity today? Draw the time line.

\[ PV_{perpetuity} = \frac{\text{Perpetuity}}{\text{rate}} \]
Suppose the perpetuity payments in the previous problem start 9 years from today. What is the value of the cash flows now? Draw the time line.

\[
\begin{align*}
N &= \\
I &= \\
Cpt \ PV &= 784,265.46
\end{align*}
\]

Pmt =
Cpt FV =

Annuities

An ordinary annuity is constant cash flow that occurs at constant and even time intervals for a finite time. The cash flows are at the end of the time period.

PV of an Annuity

Suppose you are offered $10,000 per year for three years. If the interest rate is 10 percent, what is the value of the cash flows today?

We know several ways of calculating the present value. Here is one:

Let’s look at Table A3 in the 10 percent column and the 3 period row. You find 2.4868. You can see that another way to calculate the PV of an annuity is to multiply the factor from Table A3 by the annuity.
The Equation Approach

The equation for the PV of an annuity is:

\[
PV_A = A \left( \frac{1 - \left( \frac{1}{1 + r} \right)^r}{r} \right) = 10,000 \left( \frac{1 - \left( \frac{1}{1 + 0.10} \right)^3}{0.10} \right) = 24,868.52
\]

The Table Approach

\[PV_A = A(PVIFA_{r,t})\]

\[PV_A = 10,000(PVIFA_{10\%,3})\]

\[PV_A = 10,000(2.4868)\]

\[PV_A = 24,868\]

The Calculator Approach

\[N = \]

\[I = \]

\[\text{Cpt PV} = 24,868.52\]

\[\text{Pmt} = \]

\[\text{FV} = \]

PV if the interest rate is 12% = $24,018.31

PV at 8% = $25,770.97

We can use the cash flow worksheet

\[CF_0 \quad 0\]

\[CF_1 \quad 10,000 \quad F_1 \quad 3\]

\[I \quad 10\]

\[\text{Cpt NPV} \quad 24,868.52\]
Future Value of Annuities

You are planning to save $5,000 per year for the next 30 years toward retirement with the first payment occurring one year from today. If you can earn an 11 percent interest rate, what will the value of your retirement portfolio be?

The Equation Approach

\[
FV_A = A \left( \frac{r^n - 1}{r} \right) = 5,000 \left( \frac{1.11^{30} - 1}{0.11} \right) = 995,104.39
\]

The Table Approach

\[
FV_A = A(FVIFA_{r,t})
\]

\[
FV_A = 5,000(FVIFA_{11\%,30})
\]

\[
FV_A = 5,000(199.02) = 995,100
\]

The Calculator Approach

\[
N = \\
I = \\
PV = \\
Pmt = \\
\text{Cpt FV} = \$995,104.39
\]

We cannot directly use the cash flow worksheet. This calculator does not have a “NFV” function.
You will receive $50,000 per year for 10 years with the first payment occurring 7 years from today. If the appropriate interest rate is 10.5 percent, what is the value of the cash flows today?

The Equation

\[ V_0 = 50,000(PVIFA_{10.5\%,10}) \times (PVIF_{10.5\%,6}) \]

\[ \begin{align*}
N &= 16 \\
I &= 0.105 \\
\text{Cpt PV} &= 300,738.64 \\
Pmt &= \\
FV &= \\
\text{Cpt PV} &= 165,202.10 \\
\text{Cpt PV} &= \\
\text{Cpt NPV} &= 165,202.10 \\
\end{align*} \]

We can use the cash flow worksheet

\begin{align*}
\text{CF}_0 \\
\text{CF}_1 &\quad F_1 & 6 \\
\text{CF}_2 &\quad F_2 & 10 \\
\end{align*}
Annuity Payments

You want to retire in 40 years with $1.5 million. You plan to save an equal amount each year and feel you can earn an interest rate of 11 percent. How much do you have to save each year?

The Equation

\[ 1,500,000 = A(FVIFA_{40, 11\%}) \]

\[ \begin{align*} 
N &= \\
I &= \\
PV &= \\
Cpt Pmt &= 2,578.09 \\
FV &= 
\end{align*} \]

You want to buy a car that costs $29,000. You plan to put $2,000 down and finance the remainder for 5 years. The dealer offers you a loan with a 4.5 percent annual interest rate with monthly compounding. How much are your payments?

The Equation

\[ 27,000 = A(PVIFA_{5(12), 4.5\%/12}) \]

\[ \begin{align*} 
N &= \\
I &= \\
PV &= \\
Cpt Pmt &= 503.36 \\
FV &= 
\end{align*} \]
Annuity Interest Rates

You want to have $2 million when you retire in 35 years. You plan to invest $6,000 each year to fund your retirement. What interest rate do you have to earn?

The Equation

\[ 2,000,000 = A(FVIFA_{35, r}) \]

\[
\begin{array}{c}
N \\
\text{Cpt I} & 10.89\\
PV \\
Pmt \\
FV \\
\end{array}
\]

You plan to put $20,000 down on your new house. The house sells for $200,000 and you want to get a 30 year fixed rate mortgage. If the highest payment you can afford is $1,125 per month, what is the highest interest rate you could afford?

The Equation

\[ 180,000 = A(PVIFA_{30(12), r/12}) \]

\[
\begin{array}{c}
N \\
\text{Cpt I} & 0.5327\% = r/12 \rightarrow r = (0.5327)(12) = 6.39\%\\
PV \\
Pmt \\
FV \\
\end{array}
\]
Comparing Cash Flows

Suppose a quarterback and wide receiver have both signed new contracts. The quarterback’s contract calls for an immediate signing bonus of $6 million and an annual salary of $4 million for three years. The wide receiver’s contract calls for a signing bonus of $3.5 million and an annual salary of $5 million for three years. Assuming all salary payments are at the end of the year and the interest rate is 18 percent, which contract is worth more?

QB:  The Equation

\[ PV = \frac{N \times I}{(1 + I)^N} \]

Cpt PV \[ 8,697,091.72 \]

Pmt

FV

\[ PV = 8,697,091.72 + 6,000,000 = 14,697,091.72 \]

FV = $24,147,792.00

Wide receiver:  The Equation

\[ PV = \frac{N \times I}{(1 + I)^N} \]

Cpt PV \[ 10,871,364.65 \]

Pmt

FV

\[ PV = 10,871,364.65 + 3,500,000 = 14,371,364.65 \]

FV = $23,612,612.00

Difference:

Time 0 = $325,727.07

Time 3 = $535,180.00
You are trying to plan for retirement on 10 years. You currently have $200,000 in a bond account and $400,000 in a stock account. You plan to add $10,000 per year at the end of each of the next 10 years to your bond account. The stock account earns 12.5% and the bond account earns 8.5%. When you retire, you plan to withdraw an equal amount for the next 20 years (at the end of each year) and have nothing left. Additionally, when you retire you will transfer your money to an account that earns 7.25%. How much can you withdraw each year?

The Bond Account Equation

\[ V_{10}^B = \]

\[
\begin{array}{cccc}
N & N & P^m & Cpt FV \\
I & I & PV & 452,196.69 \\
Pmt & Pmt & Cpt FV & 148,350.99 \\
\end{array}
\]

Or

\[
\begin{array}{cccc}
N & N & P^m & Cpt FV \\
I & I & PV & 600,547.68 \\
Pmt & Pmt & Cpt FV & 600,547.68 \\
\end{array}
\]

The Stock Account Equation

\[ V_{10}^S = \]

\[
\begin{array}{cccc}
N & N & P^m & Cpt FV \\
I & I & PV & 1,298,928.41 \\
Pmt & Pmt & Cpt FV & 1,298,928.41 \\
\end{array}
\]

You will have $600,547.68 + 1,298,928.41 = $1,899,476.09
The Retirement Equation

$1,899,476.09 = A(PVIFA_{20,7.25\%})

\begin{align*}
N \\
I \\
PV \\
Cpt Pmt \$182,795.78 \\
FV
\end{align*}

You are planning to save for the college education of your children. They are two years apart; one will begin college in 15 years and the other will begin college in 17 years. Assume both will be on the four year plan. You estimate each child’s education will cost $23,000 per year, payable at the beginning of each school year. The annual interest rate is 6.5 percent. How much must you deposit each year in an account to fund your children’s education? You will make your last deposit when your oldest child enters college.

\begin{align*}
CF_0 \\
CF_1 & F_1 \\
CF_2 & F_2 \\
CF_3 & F_3
\end{align*}

\begin{align*}
I \\
Cpt NPV \$157,899.32 \\
N \\
I \\
PV \\
Cpt Pmt \$6,529.58 \\
FV
\end{align*}

Growing Perpetuities

You will receive perpetuity with a payment of $10,000 next year. The payment will grow by 2.5 percent per year forever. If the appropriate interest rate is 8 percent, what is the value of the cash flows today?

\[ PV = \frac{CF_i}{r - g} \rightarrow \frac{10,000}{0.08 - 0.025} = \$181,181.18 \]
Examples with Multiple Annuities

I want to withdraw $5,000 a year for the next five years and $8,000 a year for the following five years. I can earn 10.5 percent. How much do I have to invest today? I am asking for the present value of these two annuities. There are several ways to calculate this value. First, I draw the time line.

```
0  5  10
A = 5,000  r = 10.5%  A = 8,000
```

The next step is to write an equation for the present value:

\[ V_0 = 5,000(PVIFA_{5,10.5\%}) + 8,000(PVIFA_{5,10.5\%})(PVIF_{5,10.5\%}) \]

<table>
<thead>
<tr>
<th>N</th>
<th>N</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>Pmt</td>
<td>Pmt</td>
<td>FV</td>
</tr>
<tr>
<td>cpt PV $18,714.29</td>
<td>cpt PV $29,942.87</td>
<td>cpt PV $18,175.32</td>
</tr>
</tbody>
</table>

\[ V_0 = 18,714.29 + 18,175.32 \]
\[ = 36,889.61 \]
Here is another way

\[
N \\
I \\
cpt PV \\
Pmt
\]

Now my time line looks like this:

![Time line diagram]

This is the equation:

\[
V_0 = N I \text{ cpt PV} $36,889.61 \\
Pmt \\
FV
\]

The cash flow worksheet is the last way I will show you.

\[
\begin{align*}
CF_0 &= 0 \\
CF_1 &= F_1 \\
CF_2 &= F_2 \\
I \\
cpt NPV &= 36,889.61
\end{align*}
\]

Let’s look at this time line again. I could ask the following question:

I want to withdraw $5,000 a year for 5 years and have $29,942.87 left in the investment account. How much do I have to deposit today if I can earn 10.5 percent?

I know the answer is $36,889.61

Work this outside of class.
Let’s look at this problem in a different way. I will invest $36,889.61 today in an asset that will earn 10.5 percent. I plan to withdraw $5,000 a year for the next five years. How much can I withdraw each year for the next 5 years? We know the answer is $8,000 but how would we calculate it? Here is the time line:

36,889.61 =
N
I
PV
Pmt
cpt FV $29,942.87

Now I have the following timeline:

And the equation is:

I solve the equation like this:

N
I
PV
cpt Pmt $8,000
FV
I will invest $36,889.61 today in an asset that will earn 10.5 percent. I plan to withdraw $5,000 a year for the next five years and $8,000 a year after the first five year period. How many years can I withdraw the $8,000? We know the answer is 5 years but how would we calculate it? Here is the timeline:

![Timeline Diagram]

36,889.61 = 5
N
I
PV
Pmt
cpt FV $29,942.87

Now I have the following timeline:

![Timeline Diagram]

And the equation is:

I solve the equation like this:

Cpt N 5
I
PV
Pmt
I will invest $6,000 a year for the next five years and $10,000 a year for the following five years in an asset that earns 11.25 percent. How much do I have at the end of 10 years? I am asking for the future value of these two annuities. There are several ways to calculate this value. First, I draw the time line.

The next step is to write an equation for the future value:

\[
V_{10} = \frac{A_1}{I} + \frac{A_2}{I} \cdot (1 + I)^N
\]

\[
V_{10} = \frac{6,000}{I} + \frac{10,000}{I} \cdot (1 + I)^5 = 37,553.05 + 62,588.42 = 100,141.47
\]

Here is another way

\[
V_{10} = \frac{PMT}{I} \cdot \frac{(1 + I)^N - 1}{I} \cdot (1 + I)^N
\]

\[
V_{10} = \frac{6,000}{I} \cdot \frac{(1 + I)^5 - 1}{I} \cdot (1 + I)^5 + \frac{10,000}{I} \cdot \frac{(1 + I)^5 - 1}{I} \cdot (1 + I)^5 = 37,553.05 + 62,588.42 = 100,141.47
\]
The cash flow worksheet is the last way I will show you.

<table>
<thead>
<tr>
<th>CF0</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF1</td>
<td>F1</td>
</tr>
<tr>
<td>CF2</td>
<td>F2</td>
</tr>
<tr>
<td>I</td>
<td>Pmt</td>
</tr>
<tr>
<td>Cpt NPV</td>
<td>43,588.94</td>
</tr>
</tbody>
</table>

Let’s look at this problem in a different way. I want $126,583.31 in 10 years. I plan to invest $6,000 a year for the next 5 years in an asset that will earn 11.25 percent. How much must I invest each year for the last 5 years in order to meet my goal? We know the answer is $10,000 but how would we calculate it? Here is the time line:

126,583.31 =

N
I
PV
Pmt

| cpt FV | $37,553.05 | cpt FV | $126,583.31 |
Now I have the following timeline:

I want $126,583.31 in 10 years. I plan to invest $6,000 a year for the next 5 years and then $10,000 a year in an asset that will earn 11.25 percent. How many years must I invest the $10,000? We know the answer is 5 years but how would we calculate it? Here is the time line:

\[
126,583.31 = 6,000(FVIFA_{5, 11.25\%})(FVIF_{N, 11.25\%}) + 10,000(FVIFA_{N, 11.25\%})
\]
Now I have the following timeline:

And the equation is:

I solve the equation like this:

Suppose you borrow $2000 and you are going to make annual payments of $700 each year for 3 years. What is the annual rate of this loan?

You want to receive $5000 per year for the next 5 years. How much would you need to deposit today if you can earn 9 percent?

What rate would you need to earn if you only have $15,000 to deposit?

Suppose you have $15,000 to deposit and can earn 9 percent.

How many years could you receive the $5000 payment?

How much could you receive each year for 5 years?
Effective Annual Rates (EAR)

\[ \text{EAR} = \left[ 1 + \frac{\text{APR}}{m} \right]^m - 1 \]

Use the Interest Rate Conversion Worksheet

- 12% monthly compounding = 12.68%
- 12% semiannual compounding = 12.36%
- 12% daily compounding = 12.75%

EAR versus APR

Suppose you go Vito, a local loan shark, to inquire about the interest rate on loans. You are told the interest will be a 20 percent per month. If you are brave enough to ask, what APR will Vito say you are paying? What is the EAR you are paying?

\[ \text{APR} = \]

<table>
<thead>
<tr>
<th>Nom</th>
<th>Cpt Eff</th>
</tr>
</thead>
<tbody>
<tr>
<td>C/Y</td>
<td>12</td>
</tr>
</tbody>
</table>


Inflation (or Why Inflation Doesn’t Matter)

<table>
<thead>
<tr>
<th>Today</th>
<th>Interest rate</th>
<th>One year</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100</td>
<td>10%</td>
<td>$110</td>
</tr>
<tr>
<td>Hot dogs</td>
<td>$1</td>
<td>4%</td>
</tr>
<tr>
<td>You can buy</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

What was your rate of return?

The Fisher Effect

\[(1 + R) = (1 + r)(1 + h)\]

\[\frac{1 + .10}{1 + .04} - 1 = 0.057692309\]

Approximate Fisher Effect

\[R \approx r + h\]
Question

You want to buy a motorcycle one year from now, two years from now, and three years from now. The motorcycle currently costs $20,000. You can earn a 10 percent return, and the price of the motorcycle will increase at 4 percent per year. How much do you have to deposit today in order to be able to pay cash for each motorcycle?

Nominal cash flows

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>CF</th>
<th>F</th>
<th>I</th>
<th>PV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20,800</td>
<td></td>
<td>10%</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>21,632</td>
<td></td>
<td>10%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>22,497.28</td>
<td></td>
<td>10%</td>
<td></td>
</tr>
</tbody>
</table>

\[ I = 10 \]

\[ \text{Cpt NPV} = \$53,689.32 \]

Real cash flows

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>CF</th>
<th>F</th>
<th>I</th>
<th>PV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20,000</td>
<td>3</td>
<td>5.7692%</td>
<td></td>
</tr>
</tbody>
</table>

or

<table>
<thead>
<tr>
<th>Time (t)</th>
<th>CF</th>
<th>F</th>
<th>I</th>
<th>PV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20,000</td>
<td>3</td>
<td>5.7692%</td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{Cpt PV} = \$53,689.32 \]

\[ \text{Pmt} = \$20,000 \]

\[ \text{FV} = 0 \]